*[**Solution**] Theoretical Question 2*

***Motion of an Electric Dipole in a Magnetic Field***

**(1) Conservation Laws**

**(1a)** $\vec{r}_{CM} = \frac{1}{2}(\vec{r}_1 + \vec{r}_2)$, $\vec{v}_{CM} = \frac{1}{2}(\vec{v}_1 + \vec{v}_2)$, $\vec{\ell} = \vec{r}_1 - \vec{r}_2$, $\vec{u} = \dot{\vec{\ell}} = \vec{v}_1 - \vec{v}_2$.

Total force $\vec{F}$ on the dipole is

$$\vec{F} = \vec{F}_1 + \vec{F}_2 = q(\vec{E} + \vec{v}_1 \times \vec{B}) + (-q)(\vec{E} + \vec{v}_2 \times \vec{B}) = q(\vec{v}_1 - \vec{v}_2) \times \vec{B}$$

$$= q\dot{\vec{\ell}} \times \vec{B}$$

so that

$$M\dot{\vec{v}}_{CM} = q\dot{\vec{\ell}} \times \vec{B} \qquad (M = 2m) \tag{1}$$

Computing the torque for rotation around the center of mass, we obtain

$$\begin{aligned} I\dot{\vec{\omega}} &= \left(\frac{\vec{\ell}}{2}\right) \times (q\vec{v}_1 \times \vec{B}) + \left(\frac{-\vec{\ell}}{2}\right) \times (-q\vec{v}_2 \times \vec{B}) \\ &= q\vec{\ell} \times (\vec{v}_{CM} \times \vec{B}) \end{aligned} \tag{2}$$

where

$$I = \frac{1}{2}m\ell^2 \tag{3}$$

**(1b)** From *eq.*(1), we obtain the conservation law for the momentum:

$$\dot{\vec{P}} = 0, \qquad \vec{P} = M\vec{v}_{CM} - q\vec{\ell} \times \vec{B} \tag{4}$$

From *eq.*(1) and *eq.*(2), one obtains the conservation law for the energy.

$$\dot{E} = 0, \qquad E = \frac{1}{2}Mv_{CM}^2 + \frac{1}{2}I\omega^2 \tag{5}$$

**(1c)** Using *eq.*(4) and *eq.*(2),

$$\begin{aligned} \frac{d}{dt}(\vec{r}_{CM} \times \vec{P}) \cdot \hat{B} &= (\vec{v}_{CM} \times \vec{P}) \cdot \hat{B} = -q\vec{v}_{CM} \times (\vec{\ell} \times \vec{B}) \cdot \hat{B} \\ &= q(\vec{\ell} \times \vec{B}) \times \vec{v}_{CM} \cdot \hat{B} = q(\vec{\ell} \times \vec{B}) \cdot (\vec{v}_{CM} \times \hat{B}) \\ &= q\vec{\ell} \cdot (\vec{B} \times (\vec{v}_{CM} \times \hat{B})) = -q\vec{\ell} \cdot ((\vec{v}_{CM} \times \vec{B}) \times \hat{B}) \\ &= -q\vec{\ell} \times (\vec{v}_{CM} \times \vec{B}) \cdot \hat{B} \\ &= -I\dot{\vec{\omega}} \cdot \hat{B} \end{aligned}$$

we obtain the conservation law

$$\dot{J} = 0, \qquad J = (\vec{r}_{CM} \times \vec{P} + I\vec{\omega}) \cdot \hat{B} \tag{6}$$

for the component of the angular momentum along the direction of $\vec{B}$.

**(2) Motion in a Plane Perpendicular to $\vec{B}$**

**(2a)** Write

$$\vec{\ell} = \ell\{\cos\varphi(t)\,\hat{x} + \sin\varphi(t)\,\hat{y}\}, \qquad \varphi(0) = 0, \qquad \dot{\varphi}(0) = \omega_0 \tag{7}$$

Note that

$$\vec{\omega} = \dot{\varphi}\,\hat{z} \tag{8}$$

From *eq.*(4), we have

$$M\vec{v}_{CM} = \vec{P} + q\ell B(\sin\varphi\,\hat{x} - \cos\varphi\,\hat{y}) \tag{9}$$

At $t = 0$, we have $v_{CM} = 0$, $\varphi = 0$ so that

$$\vec{P} = q\ell B\,\hat{y} \tag{10}$$

Hence from *eqs.*(9) and (10) we have

$$\dot{x}_{CM} = \left(\frac{q\ell B}{M}\right)\sin\varphi, \qquad \dot{y}_{CM} = \left(\frac{q\ell B}{M}\right)(1 - \cos\varphi) \tag{11}$$

From conservation of energy, i.e. *Eq.*(5), we have

$$\frac{1}{2}I\dot{\varphi}^2 + \frac{(q\ell B)^2}{M}(1 - \cos\varphi) = \frac{1}{2}I\omega_0^2$$

$$\therefore \quad \dot{\varphi}^2 + \frac{1}{2}\omega_c^2(1 - \cos\varphi) = \omega_0^2 \tag{12}$$

where

$$\omega_c^2 = \frac{4(q\ell B)^2}{MI} = \left(\frac{2qB}{m}\right)^2 \tag{13}$$

In order to make a full turn, $\dot{\varphi}$ can not become zero so that

$$\omega_0^2 > \omega_c^2 \;\Rightarrow\; |\omega_0| > \omega_c = \frac{2qB}{m} \tag{14}$$

**(2b)** From *Eq.*(6), we have

$$x_{CM}P + I\omega = J \tag{15}$$

where $P$ is the magnitude of $\vec{P}$.

At $t = 0$, we have $J = I\omega_0$ so that

$$x_{CM}P + I\omega = I\omega_0 \tag{16}$$

From *eq.*(12), one can see that $\omega_0^2 \geq \omega^2$ so that $x_{CM} \geq 0$. Thus $x_{CM}$ reaches a maximum $d_m$ when $\omega$ takes its minimum value.

When $\omega_0 < \omega_c$, the minimum value of $\omega$ is $-\omega_0$ so that

$$d_m = \frac{2I}{P}\omega_0 = \left(\frac{m\omega_0}{qB}\right)\ell, \qquad \omega_0 < \omega_c \tag{17}$$

When $\omega_0 > \omega_c$, the minimum value of $\omega$ is $\sqrt{\omega_0^2 - \omega_c^2}$ so that

$$d_m = \left(\frac{I}{P}\right)\left(\omega_0 - \sqrt{\omega_0^2 - \omega_c^2}\right) = \frac{m}{2qB}\left(\omega_0 - \sqrt{\omega_0^2 - \omega_c^2}\right)\ell, \qquad \omega_0 > \omega_c \tag{18}$$

When $\omega_0 = \omega_c$,

$$\omega^2 = \frac{1}{2}\omega_c^2(1 + \cos\varphi) = \omega_c^2\cos^2\frac{\varphi}{2}$$

$$\therefore \quad \dot{\varphi} = \omega_c\cos\frac{\varphi}{2}$$

When $\varphi$ is close to $\pi$, let $\varphi = \pi - 2\varepsilon$ then

$$\dot{\varepsilon} = -\frac{1}{2}\omega_c\sin\varepsilon \approx -\frac{1}{2}\omega_c\varepsilon$$

$$\therefore \quad \varepsilon \sim e^{-\omega_c t/2}$$

so that it will take $t \to \infty$ for $\varepsilon \to 0$, i.e. for $\varphi$ to reach $\pi$. Hence

$$d_m = \left(\frac{I}{P}\right)\omega_c = \left(\frac{m\omega_c}{2qB}\right)\ell, \qquad \omega_0 = \omega_c \tag{19}$$

**(2c)** Tension on the rod comes from three sources:

$$\text{(i) Coulomb force} = \frac{1}{4\pi\varepsilon_0}\frac{q^2}{\ell^2} \tag{20}$$

Positive value means compression on the rod.

$$\text{(ii) Centrifugal force due to rotation of the rod} = -\frac{1}{2}m\omega^2\ell \tag{21}$$

(iii) Magnetic force on the particles due to the motion of the center of the mass

$$= q\vec{v}_{CM} \times \vec{B} \cdot (-\hat{\ell}) = q\vec{v}_{CM} \cdot \hat{\ell} \times \vec{B}$$

Taking the square of both sides of *eq.*(4) and using the initial condition for the value of $P^2$, we obtain

$$\frac{1}{2}Mv_{CM}^2 = q\ell\vec{v}_{CM} \cdot \hat{\ell} \times \vec{B} = \frac{1}{2}I(\omega_0^2 - \omega^2) \tag{22}$$

Combining the three forces, we have

$$\text{tension on the rod} = \frac{1}{4\pi\varepsilon_0}\frac{q^2}{\ell^2} - \frac{1}{2}m\ell\omega^2 + \frac{1}{4}m\ell(\omega_0^2 - \omega^2) \tag{23}$$

A positive value means compression on the rod.