Solution to Question 2: The Rail Gun

Proper Solution (taking induced emf into consideration):

(a)

Let I be the current supplied by the battery in the absence of back emf.

Let i be the induced current by back emf ε_b.

Since $\varepsilon_b = d\phi/dt = d(BLx)/dt = BLv$, $\therefore i = Blv/R$. [1]

Net current, $I_N = I - i = I - BLv/R$. [0.5]

Forces parallel to rail are:

Force on rod due to current is $F_c = BLI_N = BL(I - BLv/R) = BLI - B^2L^2v/R$. [0.5]

Net force on rod and young man combined is $F_N = F_c - mg\sin\theta$. (1)

Newton's law: $F_N = ma = m\,dv/dt$. (2) [0.5]

Equating (1) and (2), & substituting for F_c & dividing by m, we obtain the acceleration $dv/dt = \alpha - v/\tau$, where $\alpha = BIL/m - g\sin\theta$ and $\tau = mR/B^2L^2$. [0.5]

[3]

(b)(i)

Since initial velocity of rod = 0, and let velocity of rod at time t be $v(t)$, we have

$$v(t) = v_\infty\left(1 - e^{-t/\tau}\right), \tag{3}$$

where $v_\infty(\theta) = \alpha\tau = \dfrac{IR}{BL}\left(1 - \dfrac{mg}{BLI}\sin\theta\right)$. [0.5]

Let t_s be the total time he spent moving along the rail, and v_s be his velocity when he leaves the rail, i.e. [0.5]

$$v_s = v(t_s) = v_\infty\left(1 - e^{-t_s/\tau}\right). \tag{4}$$

$$\therefore t_s = -\tau\ln\left(1 - v_s/v_\infty\right) \tag{5}$$

[0.5] [1.5]

(b)(ii)

Let t_f be the time in flight:

$$t_f = \frac{2v_s\sin\theta}{g} \tag{6}$$

[0.5]

He must travel a horizontal distance w during t_f.

$$w = (v_s\cos\theta)t_f \tag{7}$$

$$t_f = \frac{w}{v_s\cos\theta} = \frac{2v_s\sin\theta}{g} \tag{8}$$

[0.5]

(from (6) & (7))

From (8), v_s is fixed by the angle θ and the width of the strait w

$$v_s = \sqrt{\frac{gw}{\sin 2\theta}}. \tag{9}$$

$$\therefore t_s = -\tau\ln\left(1 - \frac{1}{v_\infty}\sqrt{\frac{gw}{\sin 2\theta}}\right),$$

(Substitute (9) in (5))

And

$$t_f = \frac{2\sin\theta}{g}\sqrt{\frac{gw}{\sin 2\theta}} = \sqrt{\frac{2w\tan\theta}{g}}$$

(Substitute (9) in (8)) [0.5] [1.5]

(c)

Therefore, total time is:

$$T = t_s + t_f = -\tau \ln\left(1 - \frac{1}{v_\infty}\sqrt{\frac{gw}{\sin 2\theta}}\right) + \sqrt{\frac{2w\tan\theta}{g}}$$

The values of the parameters are: $B = 10.0\,\mathrm{T}$, $I = 2424\,\mathrm{A}$, $L = 2.00\,\mathrm{m}$, $R = 1.0\,\Omega$, $g = 10\,\mathrm{m/s^2}$, $m = 80\,\mathrm{kg}$, and $w = 1000\,\mathrm{m}$.

Then

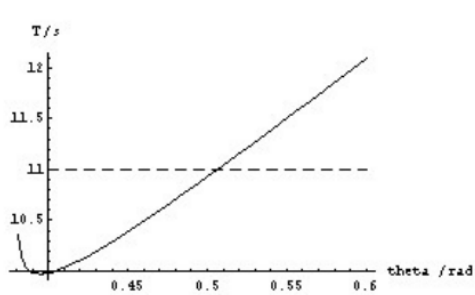
$$\tau = \frac{mR}{B^2L^2} = \frac{(80)(1.0)}{(10.0)^2(2.00)^2} = 0.20\,\mathrm{s}.$$

$$v_\infty(\theta) = \frac{2424}{(10.0)(2.00)}\left(1 - \frac{(80)(10)}{(10.0)(2.00)(2424)}\sin\theta\right) = 121(1 - 0.0165\sin\theta)$$

So,

$$T = t_s + t_f = -0.20\ln\left(1 - \frac{100}{v_\infty}\frac{1}{\sqrt{\sin 2\theta}}\right) + 14.14\sqrt{\tan\theta}$$

By plotting T as a function of θ, we obtain the following graph:



Note that the lower bound for the range of θ to plot may be determined by the condition $v_s/v_\infty < 1$ (or the argument of ln is positive), and since mg/BLI is small (0.0165), $v_\infty \approx IR/BL$ (= 121 m/s), we have the condition $\sin(2\theta) > 0.68$, i.e. $\theta > 0.37$. So one may start plotting from $\theta = 0.38$.

From the graph, for θ within the range (∼0.38, 0.505) radian the time T is within 11 s.

Labeling: 0.1 each axis
Unit: 0.1 each axis
Proper Range in θ:
0.3 lower limit (more than 0.37, less than 0.5),
0.2 upper limit (more than 0.5 and less than 0.6)
Proper shape of curve: 0.2
Accurate intersection at $\theta = 0.5$: 0.4

[1.5]

(d)

However, there is another constraint, i.e. the length of rail D. Let D_s be the distance travelled during the time interval t_s

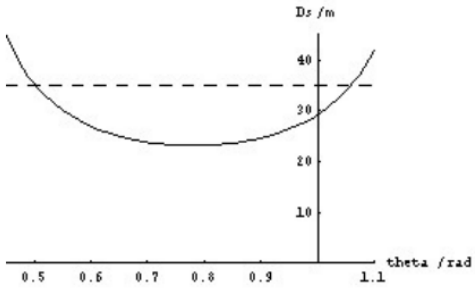
$$D_s = \int_0^{t_s} v(t)\,dt = v_\infty \int_0^{t_s}\left(1 - e^{-t/\tau}\right)dt = v_\infty\left(t + \tau e^{-\beta t}\right)\Big|_0^{t_s} = v_\infty\left[t_s - \tau\left(1 - e^{-\beta t}\right)\right] =$$

i.e.

$$D_s = -\tau \left[v_\infty(\theta) \ln \left(1 - \frac{1}{v_\infty(\theta)} \sqrt{\frac{gw}{\sin 2\theta}} \right) + \sqrt{\frac{gw}{\sin 2\theta}} \right]$$

[0.5]

The graph below shows D_s as a function of θ.



It is necessary that $D_s \leq D$, which means θ must range between .5 and1.06 radians.

In order to satisfy both conditions, θ must range between 0.5 & 0.505 radians.

(Remarks: Using the formula for t_f, t_s & D, we get

At $\theta = 0.507$, $t_f = 10.540$, $t_s = 0.466$, giving $T = 11.01$ s, & $D = 34.3$ m
At $\theta = 0.506$, $t_f = 10.527$, $t_s = 0.467$, giving $T = 10.99$ s, & $D = 34.4$ m
At $\theta = 0.502$, $t_f = 10.478$, $t_s = 0.472$, giving $T = 10.95$ s, & $D = 34.96$ m
At $\theta = 0.50$, $t_f = 10.453$, $t_s = 0.474$, giving $T = 10.93$ s, & $D = 35.2$ m,
So the more precise angle range is between 0.502 to 0.507, but students are not expected to give such answers.

To 2 sig fig $T = 11$ s. Range is 0.50 to 0.51 (in degree: 28.6° to 29.2° or 29°)

Labeling: 0.1 each axis
Unit: 0.1 each axis
Proper Range in θ:
0.3 lower limit (more than 0.4, less than 0.49),
0.2 upper limit (more than 0.51 and less than 1.1)
Proper shape of curve: 0.2
Accurate intersection at $\theta = 0.5$: 0.4

[0.5] [2.5]

Alternate Solution (Not taking induced emf into consideration):

If induced emf is not taken into account, there is no induced current, so the net force acting on the combined mass of the young man and rod is

$$F_N = BIL - mg \sin\theta.$$

[0.2 BIL; 0.2 $mg \sin\theta$]

And we have instead

$$dv/dt = \alpha,$$

where $\alpha = BIL/m - g \sin\theta$. [0.1]

$$\therefore v(t) = \alpha t$$

and

$$\therefore v_s = v(t_s) = \alpha t_s$$

[0.2]

$$t_f = \frac{2v_s \sin\theta}{g} = \frac{2\alpha t_s \sin\theta}{g}.$$

Therefore,

$$w = (v_s \cos\theta) t_f = \frac{\alpha^2 t_s^2 \sin 2\theta}{g},$$

giving

$$t_s = \frac{1}{\alpha}\sqrt{\frac{gw}{\sin 2\theta}}$$

[0.5]

and

$$t_f = \sqrt{\frac{2w\tan\theta}{g}}.$$

[0.5]

Hence,

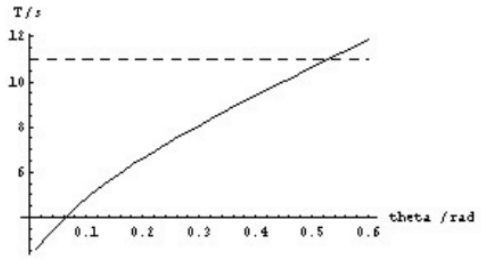
$$T = t_s + t_f = \frac{1}{\alpha}\sqrt{\frac{gw}{\sin 2\theta}} + \sqrt{\frac{2w\tan\theta}{g}} = \frac{\sqrt{wg}}{\alpha}\,\frac{\left[1 + 2\left(\frac{\alpha}{g}\right)\sin\theta\right]}{\sqrt{\sin 2\theta}}.$$

where $\alpha = BIL/m - g\sin\theta$.

The values of the parameters are: $B = 10.0\,\mathrm{T}$, $I = 2424\,\mathrm{A}$, $L = 2.00\,\mathrm{m}$, $R = 1.0\,\Omega$, $g = 10\,\mathrm{m/s^2}$, $m = 80\,\mathrm{kg}$, and $w = 1000\,\mathrm{m}$. Then,

$$T = \frac{100\,[1 + 0.20\alpha\sin\theta]}{\alpha\sqrt{\sin 2\theta}}$$

where $\alpha = 606 - 10\sin\theta$. [0.3] [2]



Labeling: 0.1 each axis
Unit: 0.1 each axis
Proper Range in θ:
0.1 lower limit (more than 0, less than 0.5),
0.2 upper limit (more than 0.52 and less than 0.8)
Proper shape of curve: 0.2
Accurate intersection at $\theta = 0.52$: 0.4

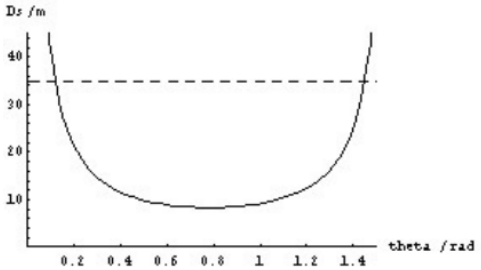
[1.3]

For θ within the range (~0, 0.52) radian the time T is within 11 s.

However, there is another constraint, i.e. the length of rail D. Let D_s be the distance travelled during the time interval t_s

$$D_s = \frac{gw}{2\alpha\sin 2\theta} = \frac{5000}{\alpha\sin 2\theta}$$

which is plotted below



Labeling: 0.1 each axis
Unit: 0.1 each axis
Proper Range in θ:
0.1 lower limit (more than 0.08, less than 0.11),
0.1 upper limit (more than 0.52 and less than 1.5)
Proper shape of curve: 0.2
Accurate intersection at $\theta = 0.11$: 0.4

[1.2]

It is necessary that $D_s \leq D$, which means θ must range between 0.11 and 1.43 radians.

In order to satisfy both conditions, θ must range between 0.11 & 0.52 radians. [0.5]