Solution and Marking Scheme

Theory

I. Satellite's orbit transfer

a) (1 point)

$$\frac{mu_0^2}{R_0} = \frac{GMm}{R_0^2}, \qquad u_0 = \sqrt{\frac{GM}{R_0}}$$

b)

conservation of angular momentum: $mu_1R_0 = mu_2R_1$

conservation of energy: $\frac{1}{2}mu_2^2 - \frac{GMm}{R_1} = \frac{1}{2}mu_1^2 - \frac{GMm}{R_0}$

$$\left[\left(\frac{R_0}{R_1}\right)^2 - 1\right] u_1^2 = 2GM\left[\frac{1}{R_1} - \frac{1}{R_0}\right]$$

$$\frac{(R_0 - R_1)(R_0 + R_1)}{R_1^2}\, u_1^2 = (2GM)\,\frac{(R_0 - R_1)}{R_0R_1}$$

$$u_1 = \sqrt{\frac{GM}{R_0}}\sqrt{\frac{2R_1}{R_1 + R_0}} = u_0\sqrt{\frac{2R_1}{R_1 + R_0}}$$

(2 points)

c) (1 point)

$$\lim_{R_1 \to \infty} u_1 = \sqrt{2}\, u_0$$

d) (1 point)

$$u_2 = u_1\frac{R_0}{R_1} = u_0\frac{\sqrt{2}\, R_0}{\sqrt{R_1(R_1 + R_0)}}$$

e)

$$u_3 = \sqrt{\frac{GM}{R_1}} = \sqrt{\frac{GM}{R_0}}\sqrt{\frac{R_0}{R_1}} = u_0\sqrt{\frac{R_0}{R_1}}$$

$$= \sqrt{\frac{R_0}{R_1}}\sqrt{\frac{R_1(R_1 + R_0)}{\sqrt{2}\, R_0}}\, u_2$$

$$u_3 = u_2\sqrt{\frac{R_1 + R_0}{2R_0}}$$

(1 point)

f) (3 points) combining equations (1) and (2):

$$\frac{d^2r}{dt^2} - \frac{C/m}{r^3} = -\frac{GM}{r^2}$$

and for the circular orbit of radius R_1 we have

$$\frac{C}{m} = GMR_1$$

hence

$$\frac{d^2r}{dt^2} - \frac{GMR_1}{r^3} = -\frac{GM}{r^2}$$

putting $r = R_1 + \eta$, where $\eta \ll R_1$

$$\therefore \quad \frac{d^2\eta}{dt^2} - \frac{GMR_1}{R_1^3\left(1 + \frac{\eta}{R_1}\right)^3} = -\frac{GM}{R_1^2\left(1 + \frac{\eta}{R_1}\right)^2}$$

$$\frac{d^2}{dt^2}\eta - \frac{GM}{R_1^2}\left(1 - 3\frac{\eta}{R_1}\right) \approx -\frac{GM}{R_1^2}\left(1 - 2\frac{\eta}{R_1}\right)$$

$$\frac{d^2}{dt^2}\eta \approx -\frac{GM}{R_1^3}\,\eta$$

the frequency of oscillation about mean distance is

$$f = \frac{1}{2\pi}\sqrt{\frac{GM}{R_1^3}}$$

the period

$$T = \frac{1}{f} = 2\pi\sqrt{\frac{R_1^3}{GM}}$$

Note that this period is the same as the orbital period

h) (1 point)

