Solution and Marking Scheme

Theory

II. Optical Gyroscope

The light wave moves with speed $c' = \frac{c}{\mu}$ in the medium having refractive index μ. Wavelength of light in medium $\lambda' = \frac{\lambda}{\mu}$, where λ is the wavelength of light in vacuum.

a) (2 points)

transit time for the CW beam:

$$t^+ = \frac{2\pi R + R\Omega t^+}{c'} = \frac{2\pi R}{c'}\left(1 - \frac{R\Omega}{c'}\right)^{-1}$$

transit time for the CCW beam:

$$t^- = \frac{2\pi R - R\Omega t^-}{c'} = \frac{2\pi R}{c'}\left(1 + \frac{R\Omega}{c'}\right)^{-1}$$

the time difference between t^+ and t^-:

$$\Delta t = \frac{4\pi R^2 \Omega}{(c')^2 - R^2\Omega^2}$$

since $(R\Omega)^2 << (c')^2$ $\qquad \Delta t \approx \frac{4\pi R^2\Omega}{(c')^2}$

b) (2 points) the round-trip optical path difference, ΔL, is given by

$$\Delta L = c'\Delta t = \frac{4\pi R^2\Omega}{c'}$$

c) (1 point) $\qquad \Delta L \cong 4.5\,\text{x}10^{-12}\,\text{m}.$

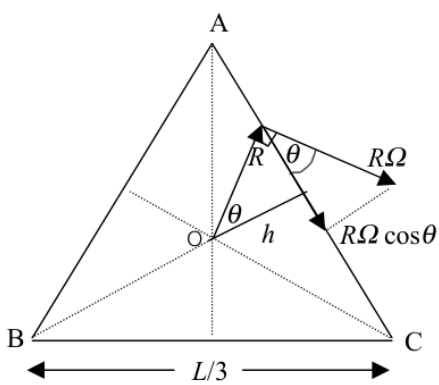
d) (1 point) the corresponding optical phase difference $\Delta\theta$ is,

$$\Delta\theta = \frac{2\pi\Delta L}{\lambda'} = \frac{8\pi^2 R^2\Omega}{c\lambda'}, \quad \text{where } \lambda' = \frac{\lambda}{\mu}$$

for N turns of fiber optic ring,

$$\Delta\theta = \frac{8\pi^2 R^2 N\Omega}{c\lambda'}$$

e) (2 points)



The figure shows the triangular ring rotating about the centre o with the angular speed Ω in the clockwise direction. Without loosing generality, let's first consider the velocity of light along AC in the CW and CCW direction,

$$v_\pm = c \pm R\Omega\cos\theta = c \pm \Omega h, \quad \text{where } h \text{ is constant.}$$

$$\tau_{\pm} = \frac{L/3}{v_{\pm}} = \frac{L/3}{c \pm \Omega h} \approx \frac{L/3}{c}\left(1 \mp \frac{\Omega h}{c}\right)$$

where $\tau_{\pm}$ is the time taken for light travelling along AC in the CW and CCW.

$$t_{\pm} = \frac{L}{v_{\pm}} = \frac{L}{c \pm \Omega h} \approx \frac{L}{c}\left(1 \mp \frac{\Omega h}{c}\right)$$

where L is the perimeter of the triangular ring.

Therefore, the time difference of light travelling in one complete cycle.

$$\Delta t = \frac{2\Omega L h}{c^2} = \frac{4\Omega}{c^2}\left(\frac{1}{2}Lh\right) = \frac{4\Omega A}{c^2}, \quad \text{where } A \text{ is the area of the triangular ring.}$$

f) The resonance frequencies associated with $L_{\pm}$ corresponding to the effective cavity lengths seen by CW and CCW propagating beams respectively is,

$$L_{+} = ct^{+} \approx L\left(1 - \frac{\Omega h}{c}\right)$$

$$L_{-} = ct^{-} \approx L\left(1 + \frac{\Omega h}{c}\right)$$

where $L_{\pm}$ is the perimeter of the equilateral triangle in the CW (+) and CCW (-) and we also use the fact that $h\Omega << c$. Therefore,

$$\Delta L = L_{-} - L_{+} = 2L\frac{\Omega h}{c} = \frac{4\Omega A}{c} = \frac{\Omega L^2}{\sqrt{3}\,c}$$

The condition to sustain the laser oscillation (given in the problem),

$$\nu_{\pm} = \frac{m}{L_{\pm}}\,c, \quad m = 1, 2, 3, \ldots \text{ integers}$$

(1 point)

$$\Delta\nu = \nu_{-} - \nu_{+} = \frac{m}{L_{-}}c - \frac{m}{L_{+}}c \approx mc\frac{\Delta L}{L^2} = \nu\frac{\Delta L}{L}$$

(1 point)

the approximation arises from $L_{+}L_{-} \approx L^2$

where L is the perimeter of the triangular ring. Hence,

$$\Delta\nu = \frac{\Delta L}{L}\nu = \frac{4A}{Lc}\nu\Omega = \frac{1}{\sqrt{3}}\frac{L}{\lambda}\Omega$$

(1 point)