Solutions of Problem No. 1
Optical fiber

1.a. At both sides of the point O (outside and inside the fiber), according to Snell law, we have:

$$n_0 \sin\theta_i = n_1 \sin\theta_1 \tag{1}$$

where θ_1 is the value of angle θ at point O *inside* the fiber.

The light trajectory lays in the xOz plane. Because the refraction index n varies along x direction, we divide Ox axis into small elements dx, so that in each of these elements n can be considered as constant. We have, then:

$$n \sin i = (n + dn) \cdot \sin(i + di) \tag{2}$$

where i is the angle between the light trajectory and x direction. Because $\theta + i = \frac{\pi}{2}$, then

$$n \cos\theta = (n + dn) \cdot \cos(\theta + d\theta) \tag{3}$$

Thus, at each point of coordinate x on the light trajectory, we have:

$$n \cos\theta = n_1 \sqrt{1 - \alpha^2 x^2}\, \cos\theta = n_1 \cos\theta_1 \tag{4}$$

Because

$$\cos\theta_1 = \sqrt{1 - \sin^2\theta_1} = \sqrt{1 - \frac{\sin^2\theta_i}{n_1^2}} \tag{5}$$

we have

$$n \cos\theta = n_1 \cos\theta_1 = n_1 \sqrt{1 - \frac{\sin^2\theta_i}{n_1^2}} = \sqrt{n_1^2 - \sin^2\theta_i}$$

Then

$$n \cos\theta = C = \sqrt{n_1^2 - \sin^2\theta_i} \tag{6}$$

1.2. Because $\dfrac{dx}{dz} = x' = \tan\theta$, from (6) we have:

$$n_1 \sqrt{1 - \alpha^2 x^2}\, \cos\theta = n_1 \sqrt{1 - \alpha^2 x^2}\, \left(1 + \tan^2\theta\right)^{-\frac{1}{2}} = C \tag{7}$$

Squaring the two sides, we obtain:

$$\left(1 - \alpha^2 x^2\right) \left(1 + \tan^2\theta\right)^{-1} = \frac{C^2}{n_1^2}$$

and

$$1 + x'^2 = \left(1 - \alpha^2 x^2\right) \frac{n_1^2}{C^2} \tag{8}$$

After derivating the two sides of (8) versus z, we get:

$$x'' + \frac{\alpha^2 n_1^2}{C^2}\, x = 0 \tag{9}$$

Because $n = n_1\sqrt{1 - \alpha^2 x^2}$ and

- $n = n_1$ at $x = 0$
- $n = n_2$ at $x = a$

we get

$$\alpha = \frac{\sqrt{n_1^2 - n_2^2}}{a \cdot n_1}$$

Finally, we get the equation for x''

$$x'' + \frac{n_1^2 - n_2^2}{a^2\left(n_1^2 - \sin^2\theta_i\right)} \cdot x = 0 \tag{10}$$

1.c. The equation for the light trajectory is obtained by solving (10). This is an equation similar to that for an harmonic oscillation, which solution can be written right away

$$x = x_0 \sin(pz + q) \tag{11}$$

with

$$p = \frac{1}{a}\sqrt{\frac{n_1^2 - n_2^2}{n_1^2 - \sin^2\theta_i}}$$

The parameters p and q are determined from the boundary conditions:

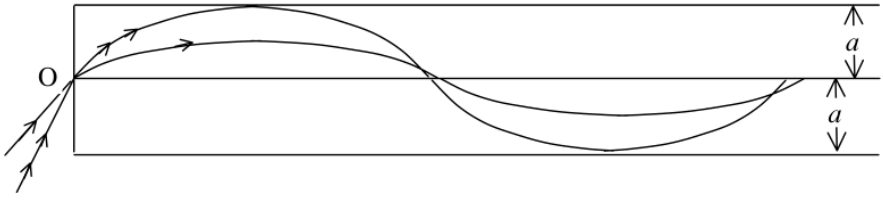
- at $z = 0$, $x = 0$, hence $q = 0$
- at $z = 0$ inside the fiber, $x' = \dfrac{dx}{dz} = \tan\theta_1$, then

$$x_0 = \frac{\tan\theta_1}{p} = \frac{a \cdot \sin\theta_i}{\sqrt{n_1^2 - n_2^2}} \tag{12}$$

The equation for the trajectory of the light inside the fiber is:

$$x = \frac{a\sin\theta_i}{\sqrt{n_1^2 - n_2^2}} \cdot \sin\left(\sqrt{\frac{n_1^2 - n_2^2}{n_1^2 - \sin^2\theta_i}} \cdot \frac{z}{a}\right) \tag{13}$$

1.d. Here is a sketch of the trajectories of two rays entering the fiber at O, under different incident angles.



2.a. The condition for the light to propagate along the fiber is that $x_0 \le a$. This means that:

$$\frac{a\sin\theta_i}{\sqrt{n_1^2 - n_2^2}} \le a$$

or:

$$\sin\theta_i \le \sqrt{n_1^2 - n_2^2} \tag{14}$$

Thus the incident angle θ_i must not exceed θ_{iM}, with

$$\sin\theta_{\mathrm{iM}} = \sqrt{n_1^2 - n_2^2} = 0.344 \tag{14a}$$

or:

$$\theta_i \le \theta_{\mathrm{iM}} = \mathrm{Arc}\sin\left(\sqrt{n_1^2 - n_2^2}\right) = \mathrm{Arc}\sin 0.344 = 0.351\,\mathrm{rad} = 20.13^\circ$$

2.b. The crossing points of the light beam with Oz axis must satisfy the condition $pz = k\pi$, with k – an integer. The z coordinates of these points are:

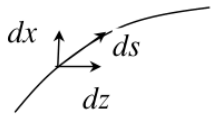
$$z = \frac{k\pi}{p} = k\pi a\sqrt{\frac{n_1^2 - \sin^2\theta_i}{n_1^2 - n_2^2}} \tag{15}$$

except for $\theta_i = 0$.

3.a. The rays entering the fiber at different incident angles have different trajectories. As a consequence, the propagation speeds of the rays along the fiber should be different.

The light trajectories are sinusoidal as given in (13). Let us calculate the time τ it takes the light to propagate from point O to its first crossing point with Oz axis. This is twice the time it takes the light to propagate from point O to its position most distant from Oz axis.

The time required for the light to travel a small segment ds along its trajectory is



$$dt = \frac{n}{c}\,ds = \frac{n}{c}\sqrt{dx^2 + dz^2} = \frac{n}{c}\sqrt{1 + \left(\frac{dz}{dx}\right)^2}\cdot dx$$
$$= \frac{n}{c}\sqrt{1 + \left(\frac{1}{\tan\theta}\right)^2}\cdot dx = \frac{n}{c}\,\frac{dx}{\sin\theta}$$

From (6), we have

$$dt = \frac{n_1^2\left(1-\alpha^2x^2\right)}{c\cdot\sqrt{\sin^2\theta_i - n_1^2\alpha^2x^2}}\cdot dx$$

and

$$\frac{\tau}{2} = \int_0^{x_0} dt = \frac{n_1^2}{c}\left[\int_0^{x_0}\frac{dx}{\sqrt{\sin^2\theta_i - n_1^2\alpha^2x^2}} - \alpha^2\int_0^{x_0}\frac{x^2\,dx}{\sqrt{\sin^2\theta_i - n_1^2\alpha^2x^2}}\right] \tag{16}$$
$$= \frac{n_1^2}{c}\left[I_1 - \alpha^2 I_2\right]$$

where

$$I_1 = \frac{1}{n_1\alpha}\operatorname{Arc\,sin}\frac{n_1\alpha x}{\sin\theta_i}\bigg|_0^{x_0} = \frac{\pi a}{2\sqrt{n_1^2 - n_2^2}} \tag{17}$$

$$I_2 = \left[-\frac{x\sqrt{\sin^2\theta_i - n_1^2\alpha^2x^2}}{2n_1^2\alpha^2}\right]_0^{x_0} + \left[\frac{\sin^2\theta_i\cdot\operatorname{Arc\,sin}\dfrac{n_1\alpha x}{\sin\theta_i}}{2n_1^3\alpha^3}\right]_0^{x_0} = \frac{\pi\sin^2\theta_i}{4n_1^3\alpha^3} \tag{18}$$

Using (16), (17), (18), we obtain

$$\tau = \frac{\pi a\cdot n_1^2}{c\sqrt{n_1^2 - n_2^2}}\left(1 - \frac{\sin^2\theta_i}{2n_1^2}\right) \tag{19}$$

The propagation speed along the fiber is $v = \dfrac{z}{\tau}$, where z is the coordinate of the first crossing point, which is determined by (15) for $k = 1$. Because z and τ depend on the incident angle θ_i, v also depends on θ_i.

For $\theta_i = \theta_{\mathrm{iM}}$, from (14a), we get

$$v_{\mathrm{M}} = \frac{\pi a n_2}{\sqrt{n_1^2 - n_2^2}}\cdot\frac{2c\sqrt{n_1^2 - n_2^2}}{\pi a n_1^2}\left(1 + \frac{n_2^2}{n_1^2}\right)^{-1}$$
$$= \frac{2cn_2}{n_1^2 + n_2^2} \tag{20}$$

and

$$v_{\mathrm{M}} = \frac{2\times 2.998\times 10^8\times 1.460}{1.500^2 + 1.460^2} = 1.998\times 10^8\,\mathrm{m/s} \tag{20a}$$

The propagation speed of the light along the Oz axis is

$$v = \frac{c}{n_1} \tag{21}$$

because the refraction index is n_1 on the axis of the fiber.

The numerical value is

$$v_0 = \frac{2.998\times 10^8}{1.5} = 1.999\times 10^8\,\mathrm{m/s} \tag{21a}$$

3.b. If the beam of the light pulses is formed by rays converging at O, then the rays with different incident angles has different propagation speeds. The two rays of incident angles $\theta_i = 0$ and $\theta_i = \theta_{\mathrm{iM}}$ arrive to the plane z with a time delay

$$\Delta t = \frac{z}{v_{\mathrm{M}}} - \frac{z}{v_0} = \frac{z}{c} \cdot \frac{(n_1 - n_2)^2}{2n_2} \tag{22}$$

This means that a very short light pulse becomes a pulse of finite width Δt given by (22) at the plane z. If two consecutive pulses enter the fiber with a delay greater than Δt, then at the plane z, they are separated. Hence the repetition frequency of the pulses must not exceed the maximal value:

$$f_{\mathrm{M}} = (\Delta t)^{-1} = \frac{2c \cdot n_2}{z \cdot (n_1 - n_2)^2} \tag{23}$$

If $z = 1000\,\mathrm{m}$, then

$$f_{\mathrm{M}} = \frac{2 \times 2.998 \times 10^8 \times 1.460}{1000 \times (1.500 - 1.460)^2} = 547.1\,\mathrm{MHz}$$