Solution:

a) Gas Volume

At the initial condition, the system is in equilibrium and the spring is unstreched; therefore

$$P_0 A = mg \quad \text{or} \quad P_0 = \frac{mg}{A} \tag{1}$$

The initial volume of gas

$$V_0 = \frac{nRT_0}{P_0} = \frac{nRT_0 A}{mg} \tag{2}$$

The work done by the gas from $\frac{1}{2}V_0$ to V

$$W_{gas} = \int_{V_0/2}^{V} P\,dV = \int_{V_0/2}^{V} \frac{P_0 V_0^{\gamma}}{V^{\gamma}}\,dV = \frac{P_0 V_0^{\gamma}}{1-\gamma}\left(V^{1-\gamma} - \left(\frac{V_0}{2}\right)^{1-\gamma}\right) \tag{3}$$

Equation (3) can also be obtained by calculating the internal energy change (without integration)

$$W_{gas} = -\Delta E = -nC_V\,(T - T_0{}') \tag{4}$$

where $T_0{}'$ is the temperature when the gas volume is $V_0/2$.

The change of the gravitational potential energy

$$\Delta_{PE} = mg\Delta h = mg\,\frac{V - \frac{1}{2}V_0}{A} \tag{5}$$

The change of the potential energy of the spring

$$\begin{aligned}\Delta_{spring} &= \frac{1}{2}kx^2 - \frac{1}{2}kx_0^2 \\ &= \frac{1}{2}\left(\frac{mgA}{V_0}\right)\left(\frac{V_0 - V}{A}\right)^2 - \frac{1}{2}\left(\frac{mgA}{V_0}\right)\left(\frac{V_0 - V_0/2}{A}\right)^2 \\ &= \frac{1}{2}\,\frac{mgV_0}{A}\left(1 - \frac{V}{V_0}\right)^2 - \frac{1}{8}\left(\frac{mgV_0}{A}\right)\end{aligned} \tag{6}$$

The kinetic energy

$$KE = \frac{1}{2}mv^2 = \frac{1}{2}\,m\,\frac{4gV_0}{5A} = \frac{2mgV_0}{5A} \tag{7}$$

By conservation of energy, we have

$$W_{gas} = \Delta_{PE} + \Delta_{spring} + KE \tag{8}$$

$$\frac{P_0 V_0^{\gamma}}{1-\gamma}\left(V^{1-\gamma} - \left(\frac{V_0}{2}\right)^{1-\gamma}\right) = mg\,\frac{V - \frac{V_0}{2}}{A} + \frac{1}{2}\,\frac{mgV_0}{A}\left(1 - \frac{V}{V_0}\right)^2 - \frac{1}{8}\,\frac{mgV_0}{A} + \frac{2}{5}\,\frac{mgV_0}{A} \tag{9}$$

$$\frac{mgV_0}{A(1-\gamma)}\left(\frac{V^{1-\gamma}}{V_0^{1-\gamma}} - \left(\frac{1}{2}\right)^{1-\gamma}\right) = mg\,\frac{V - \frac{V_0}{2}}{A} + \frac{mgV_0}{2A}\left(1 - \frac{V}{V_0}\right)^2 + \frac{11}{40}\,\frac{mgV_0}{A} \tag{10}$$

Let $s = V/V_0$, so the above equation becomes

$$\frac{1}{(1-\gamma)}\left(s^{1-\gamma} - \left(\frac{1}{2}\right)^{1-\gamma}\right) = \left(s - \frac{1}{2}\right) + \frac{1}{2}(1-s)^2 + \frac{11}{40} \tag{11}$$

With $\gamma = 5/3$ we get

$$0 = \frac{1}{2}s^2 + \frac{11}{40} + \frac{3}{2}\left(s^{-2/3} - \left(\frac{1}{2}\right)^{-2/3}\right) \tag{12}$$

Solving equation (12) numerically, we get

$$s_1 = 0.74 \quad \text{and} \quad s_2 = 1.30$$

Therefore $V_1 = 0.74V_0 = 0.74\,\dfrac{nRT_0A}{mg} = 1.88\text{m}^3$ *or* $V_2 = 1.30V_0$=3.31 m^3.

b) Small Oscillation (2 points)

The equation of motion when the piston is displaced by x from the equilibrium position is

$$m\ddot{x} = -kx - PA + mg \tag{13}$$

P is the gas pressure

$$P = \frac{P_0V_0^\gamma}{V^\gamma} = \frac{P_0V_0^\gamma}{(V_0 - Ax)^\gamma} = \frac{P_0}{\left(1 - \dfrac{Ax}{V_0}\right)^\gamma} \tag{14}$$

Since $Ax \ll V_0$ then we have $P \approx P_0\left(1 + \gamma\dfrac{Ax}{V_0}\right)$, therefore

$$m\ddot{x} \approx -kx - P_0A\left(1 + \gamma\frac{Ax}{V_0}\right) + mg$$

$$m\ddot{x} = -\left(k + P_0A\left(\gamma\frac{A}{V_0}\right)\right)x$$

$$m\ddot{x} = -\left(\frac{mgA}{V_0} + \frac{mg}{A}\,A\left(\gamma\frac{A}{V_0}\right)\right)x$$

$$m\ddot{x} + (1+\gamma)\frac{mgA}{V_0}\,x = 0 \tag{15}$$

The frequency of the small oscillation is

$$f = \frac{1}{2\pi}\sqrt{(1+\gamma)\frac{gA}{V_0}} = \frac{1}{2\pi}\sqrt{(1+\gamma)\frac{mg^2}{nRT_0}} \tag{16}$$

Numerically $f = 0.114$ Hz.

[***Marking Scheme***] *THEORETICAL Question 1A*

Spring Cylinder with Massive Piston

a. (3.0)	0.2	Initial Pressure $P_0 = mg/A$
	0.2	Initial volume $V_0 = nRT_0A/mg$
	0.3	Work done by gas $\quad W_{gas} = \dfrac{P_0V_0^\gamma}{1-\gamma}\left(V^{1-\gamma} - \left(\dfrac{V_0}{2}\right)^{1-\gamma}\right)$
	0.3	Gravitational Potential Energy $\quad \Delta_{PE} = \dfrac{mg}{A}\left(V - \dfrac{1}{2}V_0\right)$
	0.3	Spring Potential Energy $\Delta_{spring} = \dfrac{1}{2}\dfrac{mgV_0}{A}\left(1 - \dfrac{V}{V_0}\right)^2 - \dfrac{1}{8}\left(\dfrac{mgV_0}{A}\right)$
	0.3	Conservation of energy $\quad W_{gas} = \Delta_{PE} + \Delta_{spring} + KE$
	0.9(*)	Equation $\quad 0 = \dfrac{1}{2}s^2 + \dfrac{11}{40} + \dfrac{3}{2}\left(s^{-2/3} - \left(\dfrac{1}{2}\right)^{-2/3}\right)$
	0.3	$V_1 = 0.74\,V_0$ or $V_2 = 1.30\,V_0$
	0.2	$V_1 = 1.88$ m^3 or $V_2 = 3.31$ m^3
b (**2.0**)	0.5	Force Equation $\quad m\ddot{x} = -kx - PA + mg$
	0.3	Pressure $\quad P = \dfrac{P_0V_0^\gamma}{V^\gamma} = \dfrac{P_0V_0^\gamma}{(V_0 - Ax)^\gamma}$
	0.2	Approximation $\quad P \approx P_0\left(1 + \gamma\dfrac{Ax}{V_0}\right)$
	0.5	Equation $\quad m\ddot{x} + (1+\gamma)\dfrac{mgA}{V_0}x = 0$
	0.3	$f = \dfrac{1}{2\pi}\sqrt{(1+\gamma)\dfrac{mg^2}{nRT_0}}$
	0.2	$f = 0.114$ Hz

(*) Propagation errors reduce marks halved.

Solution 1 (5 points)

(1) The conservation of angular momentum (CAM) from A to B, C to D, E to F and G to H.

$$L = I\dot{\theta} = mr^2\dot{\theta} \tag{1}$$

m = mass of the child
r = distance of the child's center of mass to the swing's pivot P
$\dot{\theta}$ = the swing's angular velocity with respect to P

A to B:
Let $\dot{\theta}_d$ and $\dot{\theta}_u$ are the angular velocity at point A and B respectively, then according to CAM,

$$L_A = mr_d^2\dot{\theta}_d = L_B mr_u^2\dot{\theta}_u \tag{2}$$

so that,

$$\dot{\theta}_d = \frac{r_u^2}{r_d^2}\,\dot{\theta}_u \tag{3}$$

hence each time the swing repeat moving upward(A to B or E to F) its angular speed increases by factor of $(r_d/r_u)^2$

(2) The Conservation of Mechanical Energy (from B to C)

$$E_B = E_C = K + V = \frac{1}{2}mr_u^2\dot{\theta}_B^2 - mgr_u\,(1 - \cos\theta) \tag{4}$$

The change of the potential energy (from B to C) is the same as the rotation energy at point B,

$$mgr_u(1 - \cos\theta) = \frac{1}{2}mr_u^2\dot{\theta}_u^2 \tag{5}$$

Using the similar method, we could get the following equation for the transition from D to E,

$$mgr_d(1 - \cos\theta) = \frac{1}{2}mr_d^2\dot{\theta}_d^2 \tag{6}$$

From equations (3), (5) and (6) we have,

$$\frac{r_u}{r_d} = \left(\frac{r_u}{r_d}\right)^2 \left(\frac{\dot{\theta}_u}{\dot{\theta}_{d'}}\right)^2 \rightarrow \frac{\dot{\theta}_{d'}}{\dot{\theta}_u} = \sqrt{\frac{r_u}{r_d}} \tag{7}$$

For half a cycle we have $\dot{\theta}_{u'} = \left(\frac{r_d}{r_u}\right)^2 \dot{\theta}_{d'} = \left(\frac{r_d}{r_u}\right)^{3/2} \dot{\theta}_u$ (8)

For n complete cycles, the growth of angular velocity amplitude as well as the angular amplitude θ_A increases by a factor of $\rho_{A,n} = (r_d/r_u)^{3n}$

For $\rho_{A,n} = 2$ then with $r_d/r_u = 2^{1/10}$ one gets $\left(2^{1/10}\right)^{3n} = 2 = 2^{3n/10} \rightarrow n = \frac{10}{3}$

ALTERNATE SOLUTION

The moment of inertia with respect to the swing pivot

$$I = Mr^2 \tag{1}$$

Since the A to B transition is fast one has by conservation of angular momentum,

$$I_A\omega_A = I_B\omega_B \tag{2}$$

The energy at point A is

$$E_A = \frac{1}{2} I_A \omega_A^2 \tag{3}$$

The energy at point B is

$$E_B = \frac{1}{2} I_B \omega_B^2 + Mgh \tag{4}$$

where $h = r_d - r_u$ is the vertical distance the child's center of mass moves.

The energy at point C (conservation of energy)

$$E_C = E_B = \frac{1}{2} I_B \omega_B^2 + Mgh \tag{5}$$

As the child squats at the C to D transition, the swing losses energy of the amount Mgh so

$$E_D = \frac{1}{2} I_B \omega_B^2 \tag{6}$$

Energy at point E is equal to energy at point D (conservation energy)

$$E_E = E_D = \frac{1}{2} I_B \omega_B^2 \tag{7}$$

But we have also

$$E_E = \frac{1}{2} I_E \omega_E^2 \tag{8}$$

From equation (7) and (8) we have,

$$\omega_E^2 = \frac{I_B}{I_E}\, \omega_B^2 \tag{9}$$

Using equation(2) this equation yields,

$$\omega_E^2 = \frac{I_A}{I_B}\, \omega_A^2 \tag{10}$$

Where we have used $I_E = I_A$.
Using equation (1) one obtains from equation (10)

$$\omega_E^2 = \frac{r_d^2}{r_u^2}\, \omega_A^2 \tag{11}$$

From this one obtains,

$$\frac{|\omega_E|}{|\omega_A|} = \frac{r_d}{r_u} \tag{12}$$

This ratio gives the fractional increase in the amplitude for one half cycle of the swing motion. The fractional increase in the amplitude after n cycles is thus,

$$\frac{|\omega_E|_n}{|\omega_A|_0} = \left(\frac{r_d}{r_u}\right)^{2n} \tag{13}$$

Where $|\omega_A|_0$ is the initial amplitude and $|\omega_E|_n$ is the amplitude after n cycles. Substitute the values,

$$2 = 2^{\frac{2n}{10}} \tag{14}$$

or,

$$n = 5 \tag{15}$$

Thus it takes only 5 swing cycles for the amplitude to build up by a factor of two.

[***Marking Scheme***] *THEORETICAL Question 1B*

The Parametric Swing

(5.0)	0.25	Moment of inertia $I = Mr^2$
	0.25	Conservation of angular momentum A to B $I_A\omega_A = I_B\omega_B$
	0.25	Correct expression of energy at point A
	0.25	Correct expression of energy at point B
	0.25	Correct expression of energy at point C
	0.25	Correct expression of energy at point D
	0.25	Correct expression of energy at point E
	0.25	Conservation of angular momentum C to D
	0.50	Conservation of energy $E_B = E_C = \frac{1}{2}Mr^2\dot{\theta}^2 - Mgr + \frac{1}{2}Mg\theta^2$
	0.50	Conservation of energy $E_C = E_B = \frac{1}{2}I_B\omega_B^2 + Mgh$
	0.5	Conservation of energy $E_E = E_D = \frac{1}{2}I_B\omega_B^2 - \frac{1}{2}mr_u^2\theta^2$
	0.5	Conservation of energy $E_E = E_D = \frac{1}{2}I_B\omega_B^2$
	0.5	Equation $\frac{\lvert\omega_E\rvert}{\lvert\omega_A\rvert} = \left(\frac{r_d}{r_u}\right)^{3/2}$
	0.5	Equation $\frac{\lvert\omega_E\rvert}{\lvert\omega_A\rvert} = \frac{r_d}{r_u}$
	1.0	**Equation** $\frac{\lvert\omega_E\rvert_n}{\lvert\omega_A\rvert_0} = \left(\frac{r_d}{r_u}\right)^{3n}$
	1.0	**Equation** $\frac{\lvert\omega_E\rvert_n}{\lvert\omega_A\rvert_0} = \left(\frac{r_d}{r_u}\right)^{2n}$
	0.25	Equation $2 = 2^{\frac{3n}{10}}$
	0.25	Equation $2 = 2^{\frac{2n}{10}}$
	0.25	$n = 10/3$
	0.25	$n = 5$

Note: Propagation errors will not be considered here.