Theoretical Problem 3

3A. Average contribution of each electron to specific heat of free electron gas at constant volume (5 points)

1. According to the classical physics the conduction electrons in metals constitute free electron gas like an ideal gas. In thermal equilibrium their average energy relates to temperature, therefore they contribute to the specific heat. The average contribution of each electron to the specific heat of free electron gas at constant volume is defined as

$$c_V = \frac{d\overline{E}}{dT} \quad , \tag{1}$$

where $\overline{E}$ is the average energy of each electron. However the value of the specific heat at constant volume is a constant, independent of temperature. Please calculate $\overline{E}$ and the average contribution of each electron to the specific heat at constant volume c_V .

(1.0 points)

2. Experimentally it has been shown that the specific heat of the conduction electrons at constant volume in metals depends on temperature, and the experimental value at room temperature is about two orders of magnitude lower than its classical counterpart. This is because the electrons obey the quantum statistics rather than classical statistics. According to the quantum theory, for a metallic material the density of states of conduction electrons (the number of electronic states per unit volume and per unit energy) is proportional to the square root of electron energy E , then the number of states within energy range $\mathrm{d}E$ for a metal of volume V can be written as

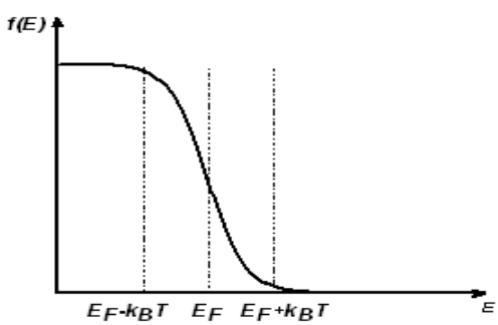
$$\mathrm{d}S = CVE^{1/2}\mathrm{d}E \tag{2}$$

where C is the normalization constant, determined by the total number of electrons of the system.

The probability that the state of energy E is occupied by electron is

$$f(E) = \frac{1}{1+\exp(\frac{E-E_F}{k_BT})} \quad , \tag{3}$$

where $k_B = 1.381 \times 10^{-23}\,\mathrm{J\cdot K^{-1}}$ is the Boltzmann constant and T is the absolute temperature, while E_F is called Fermi level. Usually at room temperature E_F is about several eVs for metallic materials (1eV= 1.602×10^{-19} J). $f(E)$ is called Fermi distribution function shown schematically in the figure below.



(1) Please calculate c_V at room temperature according to $f(E)$ (3.5points)

(2) Please give a reasonable explanation for the deviation of the classical result from that of quantum theory. (0.5 points)

Note: In your calculation the variation of the Fermi level E_F with temperature could be neglected, i.e. assume $E_F = E_F^0$, E_F^0 is the Fermi level at 0K. Meanwhile the Fermi distribution function could be simplified as a linearly descending function within an

energy range of $2k_BT$ around E_F , otherwise either 0 or 1, i.e.

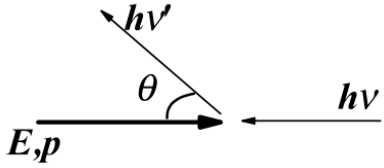
$$f(E) = \left\{ \begin{array}{cc} 1 & E < E_F - k_BT \\ \text{linearly descending function} & E_F - k_BT < E < E_F + k_BT \\ 0 & E > E_F + k_BT \end{array} \right\}$$

At room temperature $k_BT << E_F$, therefore calculation can be simplified accordingly. Meanwhile, the total number of electrons can be calculated at 0K.

3B. The Inverse Compton Scattering (5 points)

By collision with relativistic high energy electron, a photon can get energy from the high energy electron, i.e. the energy and frequency of the photon increases because of the collision. This is so-called inverse Compton scattering. Such kind of phenomenon is of great importance in astrophysics, for example, it provides an important mechanism for producing X rays and γ rays in space.

1. A high energy electron of total energy E (its kinetic energy is higher than static energy) and a low energy photon (its energy is less than the static energy of an electron) of frequency ν move in opposite directions, and collide with each other. As shown in the figure below, the collision scatters the photon, making the scattered photon move along the direction which makes an angle θ with its original incident direction (the scattered electron is not shown in the figure). Calculate the energy of the scattered photon, expressed in terms of E, ν, θ and static energy E_0 of the electron. Show the value of θ , at which the scattered photon has the maximum energy, and the value of this maximum energy. (2.4 points)



2. Assume that the energy E of the incident electron is much higher than its static energy E_0, which can be shown as $E = \gamma E_0, \gamma >> 1$, and that the energy of the incident photon is much less than E_0/γ, show the approximate expression of the maximum energy of the scattered photon. Taking $\gamma = 200$ and the wavelength of the incident visible light photon $\lambda = 500$nm, calculate the approximate maximum energy and the corresponding wavelength of the scattered photon.

Parameters: Static energy of the electron $E_0 = 0.511$MeV , Planck constant $h = 6.63 \times 10^{-34}$ J $\cdot$ s, and $hc = 1.24 \times 10^3$ eV $\cdot$ nm, where c is the light speed in the vacuum. (1.2 points)

3. (1) A relativistic high energy electron of total energy E and a photon move in opposite directions and collide with each other. Show the energy of the incident photon, of which the photon can gain the maximum energy from the incident electron. Calculate the energy of the scattered photon in this case. (0.7 points)

(2) A relativistic high energy electron of total energy E and a photon, moving in perpendicular directions respectively, collide with each other. Show the energy of the incident photon, of which the photon can gain the maximum energy from the incident electron. Calculate the energy of the scattered photon in this case. (0.7 points)