The Leidenfrost Phenomenon

3.1) As given in the problem

$$\frac{d}{dz}v \quad = \quad \left(\frac{1}{\eta}\frac{dP}{dr}\right)\cdot z \qquad \ldots\ldots\ldots\ldots \text{(i)}$$

Integrating (i) with respect to z, we get

$$v\,(z) \quad = \quad \left(\frac{1}{2\eta}\frac{dP}{dr}\right)\cdot z^2 + C \qquad \ldots\ldots\ldots\ldots \text{(ii)} \qquad (0.5 \text{ point})$$

3.2) $$v\left(\frac{b}{2}\right) \quad = \quad 0 \quad = \quad \left(\frac{1}{2\eta}\frac{dP}{dr}\right)\cdot\left(\frac{b}{2}\right)^2 + C \qquad \ldots\ldots\ldots\ldots \text{(iii)}$$

$$\therefore \qquad C \quad = \quad -\frac{b^2}{8\eta}\frac{dP}{dr} \qquad (0.5 \text{ point})$$

Note that C is not a real constant; its value depends on $\dfrac{dP}{dr}$ which is a function of r.

3.3) Let Q be the volume rate of flow of the vapour through the cylindrical surface of $2\pi rb$.

$$\delta Q \quad = \quad v\,(z)\cdot 2\pi r\delta z \quad \text{where from (ii) and (iii):} \qquad (0.3 \text{ point})$$

$$v\,(z) \quad = \quad \left(\frac{1}{2\eta}\frac{dP}{dr}\right)\cdot\left[z^2 - \frac{b^2}{4}\right] \qquad \ldots\ldots\ldots\ldots \text{(iv)}$$

$$\therefore \qquad Q \quad = \quad 2\int_{z=0}^{\frac{b}{2}} v\,(z)\cdot 2\pi r dz \quad = \quad \left(\frac{2\pi r}{\eta}\frac{dP}{dr}\right)\int_{z=0}^{\frac{b}{2}}\left[z^2 - \frac{b^2}{4}\right]dz$$

$$Q \quad = \quad -\frac{\pi r b^3}{6\eta}\frac{dP}{dr} \qquad \ldots\ldots\ldots\ldots\ldots \text{(v)} \qquad (0.7 \text{ point})$$

3.4) The total rate of heat flow from the area πr^2 of the hot surface to the drop is $\dfrac{\pi r^2 \mathcal{K}\,\Delta T}{b}$. We assume that this heat goes into vaporizing the drop.

Hence $$\rho Q\ell \quad = \quad \frac{\pi r^2 \mathcal{K}\Delta T}{b} \qquad \text{and using (v) we get}$$

$$\frac{dP}{dr} \quad = \quad -\left(\frac{6\eta\mathcal{K}\Delta T}{\rho_{\mathrm{V}}\ell b^4}\right)\cdot r \qquad \ldots\ldots\ldots\ldots\ldots \text{(vi)} \qquad (0.4 \text{ point})$$

This gives $$P\,(r) \quad = \quad -\left(\frac{3\eta\mathcal{K}\Delta T}{\rho_{\mathrm{V}}\ell b^4}\right)\cdot r^2 + B \qquad (0.4 \text{ point})$$

where B is an arbitrary constant whose value can be found by applying the boundary condition $P\,(R) = P_{\mathrm{a}}$, the atmospheric pressure.

Hence $$B = P_{\mathrm{a}} + \left(\frac{3\eta\mathcal{K}\Delta T}{\rho_{\mathrm{V}}\ell b^4}\right)\cdot R^2 \qquad \ldots\ldots\ldots\ldots \text{(vii)} \qquad (0.4 \text{ point})$$

and $$P\,(r) = P_{\mathrm{a}} + \left(\frac{3\eta\mathcal{K}\Delta T}{\rho_{\mathrm{V}}\ell b^4}\right)\cdot\left(R^2 - r^2\right) \qquad \ldots\ldots\ldots\ldots \text{(viii)} \qquad (0.8 \text{ point})$$

3.5) The net force due to pressure is in the upward direction and of magnitude

$$f = \int_{r=0}^{R}\left[P\,(r) - P_{\mathrm{a}}\right]2\pi r dr \quad = \quad \frac{3\pi\eta\mathcal{K}\Delta T R^4}{2\rho_{\mathrm{V}}\ell b^4} \qquad \ldots\ldots\ldots\text{(ix)} \qquad (1.0 \text{ point})$$

The weight of the drop is $\frac{2}{3}\pi R^3\rho_0 g$, where ρ_0 is the density of liquid.

$$\therefore \qquad \frac{2}{3}\pi R^3\rho_0 g \;=\; \frac{3\pi\eta\mathcal{K}\Delta T R^4}{2\rho_{\rm V}\ell b^4}$$

$$b \;=\; \left(\frac{9\eta\mathcal{K}R\Delta T}{4\rho_0\rho_{\rm V}\ell g}\right)^{\frac{1}{4}} \qquad \text{……(x)}$$

Note that $\dfrac{3\eta\mathcal{K}\Delta T}{\rho_{\rm V}\ell b^4} \;=\; \dfrac{4}{3}\dfrac{\rho_0 g}{R}$ ……(xi) (1.0 point)

3.6) Use equations (xi) and (viii) to obtain

$$P(r) = P_{\rm a} + \left(\frac{4}{3}\frac{\rho_0 g}{R}\right)\cdot\left(R^2 - r^2\right) \qquad \text{……(xii)}$$

$$\frac{d}{dr}P(r) = -\left(\frac{8}{3}\frac{\rho_0 g}{R}\right)\cdot r \qquad \text{……(xiii)} \qquad \text{(0.8 point)}$$

Then use (v) to calculate the total mass-rate of vaporization $Q\rho_{\rm V}$ at $r = R$:

$$\begin{aligned} Q\rho_{\rm V} &= \left(\frac{2\pi b^3 R}{12\eta}\right)\left(\frac{8}{3}\frac{\rho_0 g}{R}\right)R\rho_{\rm V} \;=\; \left(\frac{4\pi\rho_{\rm V}\rho_0 g R}{9\eta}\right)b^3 \\ &= \left(\frac{4\pi\rho_{\rm V}\rho_0 g R}{9\eta}\right)\left(\frac{9\eta\mathcal{K}R\Delta T}{4\rho_0\rho_{\rm V}\ell g}\right)^{\frac{3}{4}} \\ &= \left(\frac{4\pi^4\mathcal{K}^3\rho_{\rm V}\rho_0 g\left(\Delta T\right)^3}{9\eta\ell^3}\right)^{\frac{1}{4}}\cdot R^{\frac{7}{4}} \;=\; \beta R^{\frac{7}{4}} \quad \text{…… (xiv)} \end{aligned} \qquad \text{(1.2 points)}$$

3.7) The life-time (τ) of the drop, is to be found from

$$\begin{aligned} \frac{d}{dt}\left(\frac{2}{3}\pi R^3\rho_0\right) &= -Q\rho_{\rm V} \;=\; -\beta R^{\frac{7}{4}} \\ R^{\frac{1}{4}}\frac{d}{dt}R &= -\frac{\beta}{2\pi\rho_0} \\ \int_R^0 R^{\frac{1}{4}}dR &= -\int_0^\tau \frac{\beta}{2\pi\rho_0}dt \qquad \text{(1.0 point)} \\ \tau &= \frac{8\pi\rho_0}{5\beta}R^{\frac{5}{4}} \;=\; \frac{8}{5}\left(\frac{9\eta\rho_0^3\ell^3}{4\mathcal{K}^3\rho_{\rm V}g\left(\Delta T\right)^3}\right)^{\frac{1}{4}}\cdot R^{\frac{5}{4}} \qquad \text{(1.0 point)} \end{aligned}$$

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