

Here we study a deformable lattice hanging in gravity which acts as a deformable physical pendulum. It has only one degree of freedom, i.e. only one way to deform it and the configuration is fully described by an angle α . Such structures have been studied by famous physicist James Maxwell in 19th century, and some surprising behaviors have been discovered recently.

As shown in the figure 1, N^2 identical triangular plates (red triangle) are freely hinged by identical rods and form an $N \times N$ lattice ($N > 1$). The joints at the vertices are denoted by small circles. The sides of the equilateral triangles and the rods have the same length l . The dashed lines in the figure represent four tubes; each tube confines N vertices (grey circles) on the edge and the N vertices can slide in the tube, i.e. the tube is like a sliding rail.

The four tubes are connected in a diamond shape with two angles fixed at 60° and another two angles at 120° as shown in Figure 1. Each plate has a uniform density with mass M , and the other parts of the system are massless. The configuration of the lattice is uniquely determined by the angle α , where $0^\circ \leq \alpha \leq 60^\circ$ (please see the examples of different angle α in Figure 1). The system is hung vertically like a “curtain” with the top tube fixed along the horizontal direction.

The coordinate system is shown in Figure 2. The zero level of the potential energy is defined at $y = 0$. A triangular plate is denoted by a pair of indices (m, n) , where $m, n = 0, 1, 2, \dots, N - 1$ representing the order in the x and y directions respectively. $A(m, n)$, $B(m, n)$ and $C(m, n)$ denote the positions of the 3 vertices of the triangle (m, n) . The top-left vertex, $A(0, 0)$ (the big black circle), is fixed.

The motion of the whole system is confined in the x - y plane. The moment of inertia of a uniform equilateral triangular plate about its center of mass is $I = Ml^2/12$. The free fall acceleration is g . Please use E_k and E_p to denote kinetic energy and potential energy respectively.

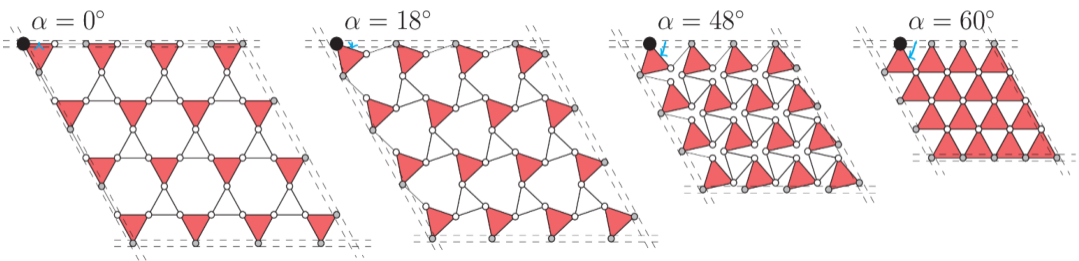


Figure 1

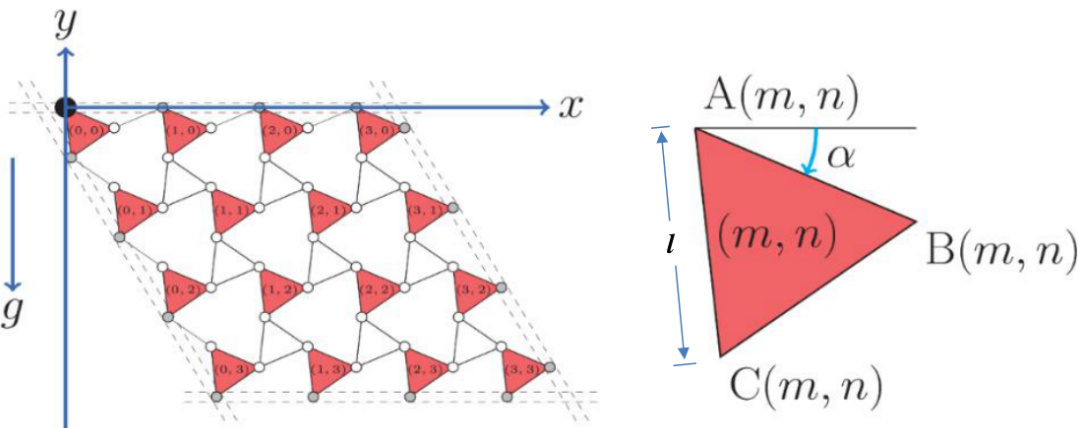


Figure 2

Section A: When $N=2$ (as shown in figure 3):

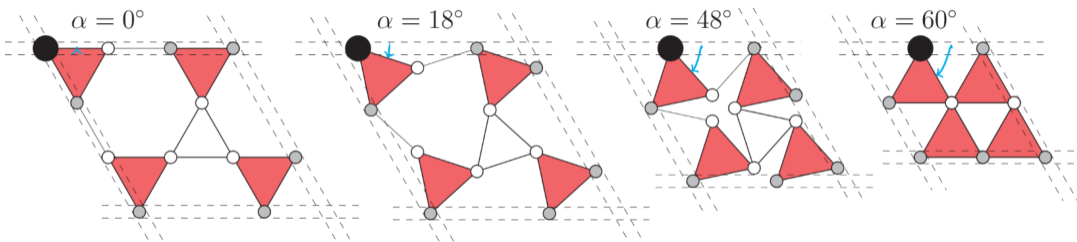


Figure 3

A1	What is the potential energy E_p of the system for a general angle α when $N = 2$?	2 points
A2	What is the equilibrium angle α_E of the system under gravity when $N = 2$?	1 point
A3	The system follows a simple harmonic oscillation under a small perturbation from equilibrium. Calculate the kinetic energy of this system in terms of $\Delta\dot{\alpha} \equiv d(\Delta\alpha)/dt$. Calculate the oscillation frequency f_E when $N = 2$.	5 points

Section B: For arbitrary N :

B1	What is the equilibrium angle α'_E under gravity when N is arbitrary?	3 points
B2	Consider the case when $N \rightarrow \infty$. Under a small perturbation of angle α , the change of potential energy of the system is $\Delta E_p \propto N^{\gamma_1}$, the kinetic energy of the system is $E_k \propto N^{\gamma_2}$, and the oscillation frequency is $f'_E \propto N^{\gamma_3}$. Find the values of γ_1 , γ_2 and γ_3 .	3 points

Section C: A force is exerted on one of the $3N^2$ triangle vertices so that the system maintains at $\alpha_m = 60^\circ$.

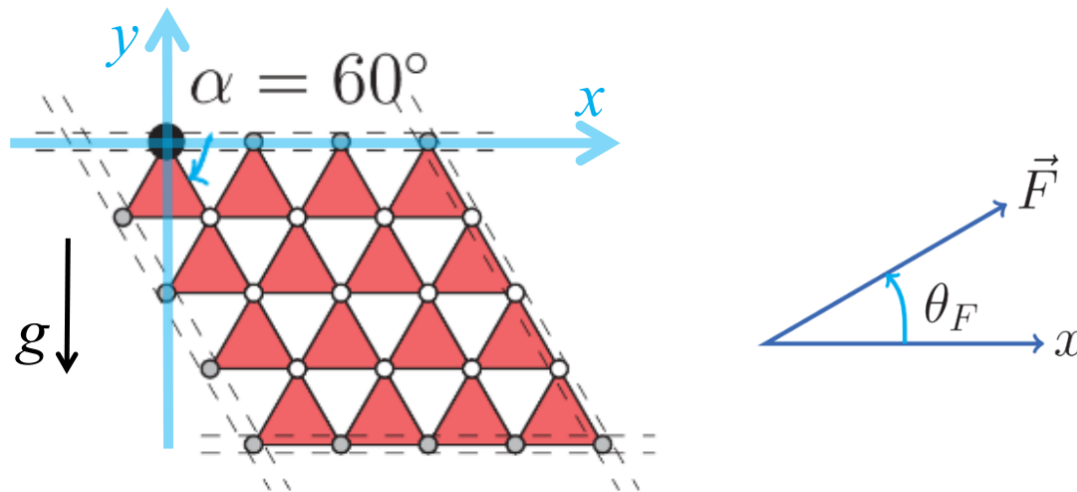


Figure 4

C1	Which vertex should we choose to minimize the magnitude of this force?	1 point
C2	What are the direction and magnitude of this minimum force? Describe the direction in terms of the angle θ_F defined in Figure 4.	5 points