

Part	Model Answer	Marks
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A1 **2**

The potential energy for $N = 2$ is:

$$E_p(\alpha) = Mg \cdot y_{c.m.}(0,0) \times 4 + Mg \cdot \Delta y \times 2 \quad \text{(0.5 points)} \quad - \text{Eq. (1)}$$

where

$$y_{c.m.}(0,0) = -\frac{\sqrt{3}l}{3} \sin\left(\frac{\pi}{6} + \alpha\right) \quad \text{(0.5 points)} \quad - \text{Eq. (2)}$$

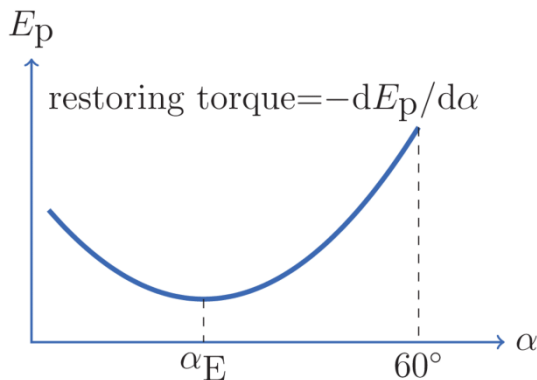
is the y coordinate of center of mass of triangle (0,0), and

$$\begin{aligned} \Delta y &= y_{A(0,1)} - y_{A(0,0)} \\ &= -l \left[\sin\left(\frac{\pi}{3} + \alpha\right) + \sin\left(\frac{\pi}{3} - \alpha\right) \right] \\ &= -\sqrt{3}l \cos \alpha \quad \text{(0.5 points)} \end{aligned} \quad - \text{Eq. (3)}$$

is the translational difference of two neighbouring triangles in y -direction. Solving Eqs. (1), (2) and (3), we obtain

$$E_p(\alpha) = -\frac{2}{3}Mgl \left(4\sqrt{3} \cos \alpha + 3 \sin \alpha \right) \quad \text{(0.5 points)} \quad - \text{Eq. (4)}$$

A2 **1**



At equilibrium, the potential energy reaches a minimum, which gives:

$$\left. \frac{dE_p(\alpha)}{d\alpha} \right|_{\alpha=\alpha_E} = 0 \quad \text{(0.5 points)} \quad - \text{Eq. (5)}$$

$$\sqrt{3} \sin \alpha_E + 3 \cos \alpha_E = 0 \quad - \text{Eq. (6)}$$

or

$$\alpha_E = \tan^{-1} \frac{\sqrt{3}}{4} \quad \text{(0.5 point)} \quad - \text{Eq. (7)}$$

A3 **5**

If the total energy of the oscillation has the following form

$$E(\Delta\alpha, \Delta\dot{\alpha}) = E_p + E_k = \frac{1}{2}K(\Delta\alpha)^2 + \frac{1}{2}I(\Delta\dot{\alpha})^2, \quad \text{(0.5 points)} \quad - \text{Eq. (8)}$$

where E_p and E_k are the potential and kinetic energies of the system respectively, then the motion is a simple harmonic oscillation with angular frequency $\omega = \sqrt{K/I}$. Here $\Delta\alpha = \alpha - \alpha_E$. Under a small perturbation, the potential energy change is:

$$\begin{aligned}\Delta E_p &\approx \frac{1}{2} \left. \frac{d^2 E_p}{d\alpha^2} \right|_{\alpha=\alpha_E} (\Delta\alpha)^2 \\ &= \left(\frac{1}{2} \right) \left(\frac{2}{3} Mgl \right) (4\sqrt{3} \cos \alpha_E + 3 \sin \alpha_E) (\Delta\alpha)^2 \\ &= \frac{\sqrt{57}}{3} Mgl (\Delta\alpha)^2 \quad \textbf{(1 point)}\end{aligned} \quad \text{- Eq. (9)}$$

The total kinetic energy of the system includes the translational kinetic energy of every plate and the rotational kinetic energy of every plate relative to its center of mass

$$E_k = \sum E_k^{\text{trans}} + \sum E_k^{\text{rot}} \quad \text{- Eq. (10)}$$

The rotational kinetic energy is

$$\sum E_k^{\text{rot}} = 4 \times \frac{1}{2} \frac{Ml^2}{12} (\Delta\dot{\alpha})^2 = \frac{1}{6} Ml^2 (\Delta\dot{\alpha})^2 \quad \textbf{(0.5 points)} \quad \text{- Eq. (11)}$$

E_k^{trans} can be obtained by considering the motion of the center of mass of each triangle and setting $N = 2$.

$$\begin{aligned}x_{\text{c.m.}(m,n)} &= m(2l \cos \alpha) + n(2l \cos \alpha) \cos \frac{\pi}{3} + \frac{l}{\sqrt{3}} \cos \left(\alpha + \frac{\pi}{6} \right), \\ y_{\text{c.m.}(m,n)} &= -n(2l \cos \alpha) \sin \frac{\pi}{3} - \frac{l}{\sqrt{3}} \sin \left(\alpha + \frac{\pi}{6} \right). \quad \textbf{(0.5 point)}\end{aligned}$$

Differentiating and substituting

$$\begin{aligned}\sin \alpha &= \frac{\sqrt{3}}{\sqrt{19}}, \cos \alpha = \frac{4}{\sqrt{19}}, \sin \left(\alpha + \frac{\pi}{6} \right) = \frac{7}{2\sqrt{19}}, \cos \left(\alpha + \frac{\pi}{6} \right) = \frac{3\sqrt{3}}{2\sqrt{19}}, \\ \dot{x}_{\text{c.m.}(m,n)} &= - \left(2m + n + \frac{7}{6} \right) \frac{3}{\sqrt{57}} l \Delta\dot{\alpha}, \quad \dot{y}_{\text{c.m.}(m,n)} = \frac{3(2n-1)}{2\sqrt{19}} l \Delta\dot{\alpha}. \\ v_{\text{c.m.}(m,n)}^2 &= \dot{x}_{\text{c.m.}(m,n)}^2 + \dot{y}_{\text{c.m.}(m,n)}^2 = \frac{(12m+6n+7)^2 + 27}{228} l^2 (\Delta\dot{\alpha})^2, \quad \textbf{(1 point)} \\ E_{\text{c.m.,k}}^{\text{trans}} &= \frac{M}{2} \left[v_{\text{c.m.}(0,0)}^2 + v_{\text{c.m.}(0,1)}^2 + v_{\text{c.m.}(1,0)}^2 + v_{\text{c.m.}(1,1)}^2 \right] = \frac{164}{57} Ml^2 (\Delta\dot{\alpha})^2. \\ E_k^{\text{trans}} &= E_{\text{c.m.,k}}^{\text{trans}} + E_k^{\text{rot}} = \frac{347}{114} Ml^2 (\Delta\dot{\alpha})^2. \quad \textbf{(1 point)}\end{aligned}$$

Alternatively, another way to get E_k^{trans} is based on the center of mass of the whole system:

$$E_k = \sum E_{\text{c.m.,k}}^{\text{trans}} + \sum E_{\text{r.c.,k}}^{\text{rot}} \quad \textbf{(0.5 points)} \quad \text{- Eq. (12)}$$

where

$$E_{\text{r.c.,k}}^{\text{trans}} = \frac{M}{2} \left[v_{\text{r.c.}(0,0)}^2 + v_{\text{r.c.}(1,0)}^2 + v_{\text{r.c.}(0,1)}^2 + v_{\text{r.c.}(1,1)}^2 \right] \quad \text{- Eq. (13)}$$

is the translational kinetic energy relative to the center of mass of the system and

$$E_{\text{c.m.,k}}^{\text{trans}} = \frac{4M}{2} v_{\text{c.m.}}^2 \quad \text{- Eq. (14)}$$

is the translational kinetic energy of the center of mass of the system.

The center of mass of each of the $2 \times 2 = 4$ triangles always form diamond shape with lateral length $2l \cos \alpha$. The center of mass of the whole system is at the center of the diamond shape. Hence

$$\begin{aligned}v_{\text{r.c.}(0,0)} &= v_{\text{r.c.}(1,1)} = \left. \frac{d(\sqrt{3}l \cos \alpha)}{d\alpha} \right|_{\alpha=\alpha_E} \Delta\dot{\alpha} \\ v_{\text{r.c.}(1,0)} &= v_{\text{r.c.}(0,1)} = \left. \frac{d(l \cos \alpha)}{d\alpha} \right|_{\alpha=\alpha_E} \Delta\dot{\alpha}\end{aligned} \quad \text{- Eq. (15)}$$

Substituting Eqs. (14) and (15) into Eq. (13), we obtain

$$E_{\text{r.c.,k}}^{\text{trans}} = 4 \sin^2 \alpha_E M l^2 (\Delta \dot{\alpha})^2 \quad - \text{Eq. (16)}$$

For $E_{\text{c.m.,k}}^{\text{trans}}$,

$$v_{\text{c.m.}} = \sqrt{\left(\frac{dx_{\text{c.m.}}}{d\alpha}\right)^2 + \left(\frac{dy_{\text{c.m.}}}{d\alpha}\right)^2} \bigg|_{\alpha=\alpha_E} \Delta \dot{\alpha} \quad - \text{Eq. (17)}$$

is the velocity of the center-of-mass of the four triangular plates, with

$$\begin{aligned} x_{\text{c.m.}} &= x_{\text{c.m.}(0,0)} + \frac{1}{2} (x_{\text{B}(0,0)} + x_{\text{A}(1,0)}) \\ &= \frac{\sqrt{3}l}{3} \cos\left(\frac{\pi}{6} + \alpha\right) + \frac{3}{2}l \cos \alpha \end{aligned} \quad - \text{Eq. (18)}$$

$$\begin{aligned} y_{\text{c.m.}} &= y_{\text{c.m.}(0,0)} + \frac{1}{2} \Delta y \\ &= -\frac{\sqrt{3}l}{3} \sin\left(\frac{\pi}{6} + \alpha\right) - \frac{\sqrt{3}}{2}l \cos \alpha \end{aligned} \quad - \text{Eq. (19)}$$

Substituting Eqs. (17), (18) and (19) and into Eq. (14), we obtain

$$E_{\text{c.m.,k}}^{\text{trans}} = \left(\frac{2}{3} + 10 \sin^2 \alpha_E\right) M l^2 (\Delta \dot{\alpha})^2 \quad \textbf{(0.5 points)} \quad - \text{Eq. (20)}$$

Combining Eqs. (12), (16) and (20), we obtain

$$\begin{aligned} E_{\text{k}} &= E_{\text{k}}^{\text{rot}} + E_{\text{r.c.,k}}^{\text{trans}} + E_{\text{c.m.,k}}^{\text{trans}} \\ &= \left(\frac{5}{6} + 14 \sin^2 \alpha_E\right) M l^2 (\Delta \dot{\alpha})^2 \\ &= \frac{347}{114} M l^2 (\Delta \dot{\alpha})^2 \quad \textbf{(1.5 points)} \end{aligned} \quad - \text{Eq. (21)}$$

According to Eqs. (8), (9) and (21),

$$f = \frac{1}{2\pi} \sqrt{\frac{\frac{\sqrt{57}}{3} M g l}{\frac{347}{114} M l^2}} = \frac{1}{2\pi} \sqrt{\frac{38\sqrt{57}}{347} \frac{g}{l}} \quad \textbf{(0.5 points)} \quad - \text{Eq. (22)}$$

[Note 1: 0.5 point should be deducted if there are numerical mistakes, but all steps are correct.]

Note 2: A rough estimate of $f \sim \sqrt{\frac{g}{l}}$ can get 0.5 points out of 5 points.]

B1

3

For arbitrary N , the total potential energy

$$E_{\text{p}} = \sum_{m,n=0}^{N-1} E_{\text{p}}(m, n) \quad - \text{Eq. (23)}$$

where

$$E_{\text{p}}(m, n) = \frac{1}{3} M g [y_{\text{A}(m,n)} + y_{\text{B}(m,n)} + y_{\text{C}(m,n)}] \quad - \text{Eq. (24)}$$

(0.5 points for Eqs. (23) and (24))

and

$$\begin{aligned} y_{\text{A}(m,n)} &= -nl \sin\left(\frac{\pi}{3} - \alpha\right) - nl \sin\left(\frac{\pi}{3} + \alpha\right) = -\sqrt{3}nl \cos \alpha \\ y_{\text{B}(m,n)} &= y_{\text{A}(m,n)} - l \sin \alpha = -\sqrt{3}nl \cos \alpha - l \sin \alpha \\ y_{\text{C}(m,n)} &= y_{\text{A}(m,n)} - l \sin\left(\frac{\pi}{3} + \alpha\right) = -\sqrt{3}nl \cos \alpha - l \sin\left(\frac{\pi}{3} + \alpha\right) \end{aligned} \quad - \text{Eq. (25)}$$

(0.5 points for all three correct coordinates)

Thus,

$$E_{\text{p}}(m, n) = -\frac{1}{3} M g l \left[3\sqrt{3}n \cos \alpha + \sin \alpha + \sin\left(\frac{\pi}{3} + \alpha\right) \right] \quad - \text{Eq. (26)}$$

and

$$E_p = \sum_{m,n=0}^{N-1} E_p(m,n) \\ = -\frac{1}{3}Mgl \sum_{m,n=0}^{N-1} [3\sqrt{3}n \cos \alpha + \sin \alpha + \sin \left(\frac{\pi}{3} + \alpha\right)] \quad \textbf{(0.5 points)} \quad - \text{Eq. (27)}$$

Using the mathematical relations

$$\sum_{m=0}^{N-1} 1 = \sum_{n=0}^{N-1} 1 = N$$

and

$$\sum_{m=0}^{N-1} m = \sum_{n=0}^{N-1} n = \frac{N(N-1)}{2} \quad - \text{Eq. (28)},$$

Eq. (27) becomes

$$E_p = -\frac{1}{3}N^2Mgl \left[\frac{3\sqrt{3}(N-1) \cos \alpha}{2} + \sin \alpha + \sin \left(\frac{\pi}{3} + \alpha\right) \right] \\ \text{or} \quad = -\frac{1}{3}N^2Mgl \left[\frac{\sqrt{3}(3N-2) \cos \alpha}{2} + \frac{3}{2} \sin \alpha \right] \quad \textbf{(1 points)} \quad - \text{Eq. (29)}$$

At equilibrium, $\frac{dE_p}{d\alpha} = 0$, therefore

$$-\frac{3\sqrt{3}(N-1) \sin \alpha'_E}{2} + \cos \alpha'_E + \cos \left(\frac{\pi}{3} + \alpha'_E\right) = 0 \quad - \text{Eq. (30)}$$

$$\alpha'_E = \tan^{-1} \left(\frac{\sqrt{3}}{3N-2} \right) \quad \textbf{(0.5 points)} \quad - \text{Eq. (31)}$$

[Remark: Increasing α lowers each triangle relative to its vertex A, but globally raises the system, i.e. the bottom tube is raised higher. When $N \rightarrow \infty$, the global displacement dominates, consequently $\alpha \rightarrow 0$.]

B2

3

Under a small perturbation, the potential energy change, according to Eq. (29) is

$$\Delta E_p \approx \frac{1}{2} \frac{d^2 E_p}{d\alpha^2} \Big|_{\alpha=\alpha'_E} (\Delta \alpha)^2 \sim N^3 \quad \text{or} \quad \gamma_1 = 3 \quad \textbf{(0.5 points)} \quad - \text{Eq. (32)}$$

[Remark: There are N^2 triangles and the y coordinate of the total center of mass is proportional to N , hence $E_p \sim N^3$ and $\gamma_1 = 3$. Using this argument to derive the correct γ_1 can also get 0.5 points.]

The kinetic energy of a triangle includes the translational energy of its center of mass and the rotational energy about its center of mass. Hence the total kinetic energy of the N^2 triangles is

$$E_k = \sum_{m,n} E_{c.m.(m,n)} + \sum_{m,n} E_{r.c.(m,n)} \quad - \text{Eq. (33)}$$

where

$$E_{r.c.(m,n)} = \frac{1}{2} \frac{Ml^2}{12} (\Delta \dot{\alpha})^2 = \frac{1}{24} Ml^2 (\Delta \dot{\alpha})^2 \sim 1 \quad - \text{Eq. (34)}$$

and

$$E_{c.m.(m,n)} = \frac{M}{2} v_{c.m.(m,n)}^2 \\ = \frac{M(\Delta \dot{\alpha})^2}{2} \left[\left(\frac{dx_{c.m.(m,n)}}{d\alpha} \right)^2 + \left(\frac{dy_{c.m.(m,n)}}{d\alpha} \right)^2 \right]_{\alpha=\alpha'_E} \quad \textbf{(0.5 points)} \quad - \text{Eq. (35)}$$

Since

$$x_{c.m.(m,n)} = x_{A(m,n)} + \frac{\sqrt{3}l}{3} \cos \left(\frac{\pi}{6} + \alpha \right) \\ = (2m+n)l \cos \alpha + \frac{l}{2} \cos \alpha - \frac{\sqrt{3}l}{6} \sin \alpha$$

and

$$\begin{aligned}
 y_{\text{c.m.}(m,n)} &= y_{\text{A}(m,n)} + \frac{\sqrt{3}l}{3} \sin\left(\frac{\pi}{6} + \alpha\right) \\
 &= \sqrt{3}nl \cos \alpha + \frac{\sqrt{3}l}{6} \cos \alpha + \frac{l}{2} \sin \alpha \quad - \text{Eq. (36)}
 \end{aligned}$$

(0.5 points for correct x and y)

$$\begin{aligned}
 \frac{dx_{\text{c.m.}(m,n)}}{d\alpha} &= \left[-(2m+n) \sin \alpha - \frac{1}{2} \sin \alpha - \frac{\sqrt{3}}{6} \cos \alpha \right] l \\
 \frac{dy_{\text{c.m.}(m,n)}}{d\alpha} &= \left[-\sqrt{3}n \sin \alpha - \frac{\sqrt{3}}{6} \sin \alpha + \frac{1}{2} \cos \alpha \right] l
 \end{aligned}$$

we have

$$E_{\text{c.m.}(m,n)} = \frac{1}{2} M l^2 (\Delta \dot{\alpha})^2 \left[(4m^2 + 4n^2 + 4mn + 2m + 2n) \sin^2 \alpha'_E + \frac{2\sqrt{3}}{3} (m-n) \sin \alpha'_E \cos \alpha'_E + \frac{1}{3} \right] \quad - \text{Eq. (37)}$$

Since $\alpha'_E \sim \frac{1}{N}$ in Eq. (31), we have

$$E_{\text{c.m.}(m,n)} = A \cdot N^2 \cdot \frac{1}{N^2} + B \cdot N \cdot \frac{1}{N} + C \sim 1 \quad \textbf{(0.5 points)} \quad - \text{Eq. (38)}$$

According to Eqs. (33), (34) and (38), we have

$$\begin{aligned}
 E_k &= \sum_{m,n} E_{\text{c.m.}(m,n)} + \sum_{m,n} E_{\text{r.c.}(m,n)} \sim N \times N \times 1 \sim N^2 \\
 &\quad \text{or } \gamma_2 = 2 \quad \textbf{(0.5 points)} \quad - \text{Eq. (39)}
 \end{aligned}$$

[Remarks: $E_k \sim N^2$ because there are N^2 triangles, each contribute $E_{\text{r.c.}(m,n)}$ (relative-to-center-of-mass kinetic energy) and $E_{\text{c.m.}(m,n)} \sim 1$ (center-of-mass kinetic energy).]

Note that $E_{\text{r.c.}(m,n)} \sim 1$ is true for arbitrary α while $E_{\text{c.m.}(m,n)} \sim 1$ is only true for the special case of $\alpha'_E \rightarrow 0$ or $N \rightarrow \infty$.

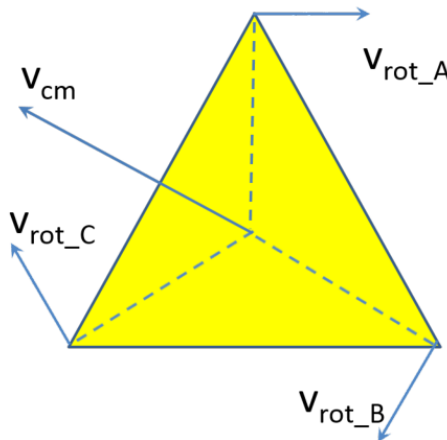
Therefore

$$\begin{aligned}
 f'_E &\sim \sqrt{\frac{E_p}{E_k}} \sim \sqrt{N} \\
 \text{or } \gamma_3 &= 0.5 \quad \textbf{(0.5 points)} \quad - \text{Eq. (40)}
 \end{aligned}$$

C1

1

The minimum force should act on the farthest triangle $(N-1, N-1)$, whose motion can be decomposed into the motion of the center of mass and the rotation around the center of mass: $\vec{v} = \vec{v}_{\text{c.m.}} + \vec{v}_{\text{rot.}}$ As shown in the figure, $\vec{v}_{\text{rot.}}$ of vertex C makes the smallest angle relative to the direction of $\vec{v}_{\text{c.m.}}$ near $\alpha_m \equiv \pi/3$. Hence its displacement is the largest and its corresponding force is minimum, i.e. the minimum force should act on vertex C $(N-1, N-1)$. **(1 point)**



At $\alpha = \alpha_m \equiv \pi/3$, a small change in α will change the potential energy by:

$$\begin{aligned}\Delta E_p(\alpha_m) &= \left. \frac{dE_p}{d\alpha} \right|_{\alpha=\alpha_m} \Delta\alpha \\ &= \frac{1}{3}N^2Mgl \left[\left(\frac{3\sqrt{3}N}{2} - \sqrt{3} \right) \sin \alpha_m - \frac{3}{2} \cos \alpha_m \right] \Delta\alpha \\ &= \frac{3}{4}(N-1)N^2Mgl\Delta\alpha \quad \textbf{(1 point)}\end{aligned} \quad \text{- Eq. (41)}$$

The displacement of $C(m,n)$ point is

$$\begin{aligned}\Delta x_{C(m,n)} &= - \left[(2m+n) \sin \alpha_m - \sin \left(\frac{\pi}{3} + \alpha_m \right) \right] l\Delta\alpha \\ &= \frac{(2m+n+1)\sqrt{3}}{2} l\Delta\alpha \quad \textbf{(0.5 points)} \\ \Delta y_{C(m,n)} &= - \left[\sqrt{3}n \sin \alpha_m - \cos \left(\frac{\pi}{3} + \alpha_m \right) \right] l\Delta\alpha \\ &= \frac{(3n+1)}{2} l\Delta\alpha \quad \textbf{(0.5 points)}\end{aligned}$$

For $C(N-1, N-1)$, $\Delta r = \sqrt{(\Delta x)^2 + (\Delta y)^2} = (3N-2)(l\Delta\alpha)$. **(1 point)**

Hence

$$F_{\min} = \frac{\Delta E_p(\alpha_m)}{\Delta r_{\max}} = \frac{3(N-1)N^2}{4(3N-2)} Mg \quad \textbf{(1 point)} \quad \text{- Eq. (42)}$$

and

$$\begin{aligned}\theta_{F_{\min}} &= \tan^{-1} \left[\frac{\Delta y_{C(N-1, N-1)}}{\Delta x_{C(N-1, N-1)}} \right] + \pi \\ &= -\tan^{-1} \frac{\sqrt{3}}{3} + \pi = \frac{5\pi}{6} \quad \textbf{(1 point)}\end{aligned} \quad \text{- Eq. (43)}$$

[Remarks: This $\theta_{F_{\min}}$ is not perpendicular to the $C(N-1, N-1)$ – $A(0,0)$ direction because of the constraints of the tunes, e.g. $A(1,0)$, $A(2,0)$, $A(3,0)$, $\dots$, are also the holding points.]

N/A

Appendix 1:

(a) Calculation of the exact E_p , E_k and f'_E in Parts (C), (D) and € for arbitrary N
Under a small perturbation, the potential energy change is

$$\begin{aligned}\Delta E_p &\approx \frac{1}{2} \left. \frac{d^2 E_p}{d\alpha^2} \right|_{\alpha=\alpha'_E} (\Delta\alpha)^2 \\ &= \frac{1}{3}N^2Mgl \left(\frac{3\sqrt{3}N - 2\sqrt{3}}{2} \cos \alpha'_E + \frac{3}{2} \sin \alpha'_E \right) \frac{(\Delta\alpha)^2}{2} \\ &= \frac{\sqrt{3(3N-2)^2+9}}{12} N^2Mgl(\Delta\alpha)^2\end{aligned} \quad \text{- Eq. (44)}$$

The kinetic energy of a triangle includes the translational energy of its center of mass and the rotational energy around its center of mass. Hence the total kinetic energy of the N^2 triangles is

$$E_k = \sum_{m,n} E_{c.m.(m,n)} + \sum_{m,n} E_{r.c.(m,n)} \quad \text{- Eq. (45)}$$

where

$$E_{r.c.(m,n)} = \frac{1}{2} \frac{Ml^2}{12} (\Delta\dot{\alpha})^2 = \frac{1}{24} Ml^2 (\Delta\dot{\alpha})^2 \quad \text{- Eq. (46)}$$

and

$$\begin{aligned}
E_{\text{c.m.}(m,n)} &= \frac{M}{2} v_{\text{c.m.}(m,n)}^2 \\
&= \frac{M(\Delta\dot{\alpha})^2}{2} \left[\left(\frac{dx_{\text{c.m.}(m,n)}}{d\alpha} \right)^2 + \left(\frac{dy_{\text{c.m.}(m,n)}}{d\alpha} \right)^2 \right]_{\alpha=\alpha'_E}
\end{aligned} \quad - \text{ Eq. (47)}$$

Since

$$\begin{aligned}
x_{\text{c.m.}(m,n)} &= x_{\text{A}(m,n)} + \frac{\sqrt{3}l}{3} \cos \left(\frac{\pi}{6} + \alpha \right) \\
&= (2m+n)l \cos \alpha + \frac{l}{2} \cos \alpha - \frac{\sqrt{3}l}{6} \sin \alpha
\end{aligned}$$

and

$$\begin{aligned}
y_{\text{c.m.}(m,n)} &= y_{\text{A}(m,n)} - \frac{\sqrt{3}l}{3} \sin \left(\frac{\pi}{6} + \alpha \right) \\
&= -\sqrt{3}nl \cos \alpha - \frac{\sqrt{3}l}{6} \cos \alpha - \frac{l}{2} \sin \alpha
\end{aligned} \quad - \text{ Eq. (48)}$$

Hence,

$$\begin{aligned}
\frac{dx_{\text{c.m.}(m,n)}}{d\alpha} &= \left[-(2m+n) \sin \alpha - \frac{1}{2} \sin \alpha - \frac{\sqrt{3}}{6} \cos \alpha \right] l \\
\frac{dy_{\text{c.m.}(m,n)}}{d\alpha} &= \left[-\sqrt{3}n \sin \alpha + \frac{\sqrt{3}}{6} \sin \alpha - \frac{1}{2} \cos \alpha \right] l
\end{aligned}$$

We have

$$E_{\text{c.m.}(m,n)} = \frac{1}{2} M l^2 (\Delta\dot{\alpha})^2 \left[(4m^2 + 4n^2 + 4mn + 2m + 2n) \sin^2 \alpha'_E + \frac{2\sqrt{3}}{3} (m-n) \sin \alpha'_E \cos \alpha'_E + \frac{1}{3} \right] \quad - \text{ Eq. (49)}$$

and

$$\begin{aligned}
E_k &= \sum_{m,n} E_{\text{c.m.}(m,n)} + \sum_{m,n} E_{\text{r.c.}(m,n)} \\
&= \left[\frac{1}{6} (11N-1)(N-1) \sin^2 \alpha'_E + \frac{5}{24} \right] N^2 M l^2 (\Delta\dot{\alpha})^2 \\
&= \left[\frac{(11N-1)(N-1)}{2(3N-2)^2+6} + \frac{5}{24} \right] N^2 M l^2 (\Delta\dot{\alpha})^2
\end{aligned} \quad - \text{ Eq. (50)}$$

With Eqs. (44) and (50), we have

$$\begin{aligned}
f'_E &= \frac{1}{2\pi} \sqrt{\frac{\frac{\sqrt{3(3N-2)^2+9}}{12} N^2 M g l}{\left[\frac{(11N-1)(N-1)}{2(3N-2)^2+6} + \frac{5}{24} \right] N^2 M l^2}} \\
&= \frac{1}{2\pi} \sqrt{\frac{2\sqrt{3(3N-2)^2+9}}{\left[\frac{12(11N-1)(N-1)}{(3N-2)^2+3} + 5 \right]}} \frac{g}{l}
\end{aligned} \quad - \text{ Eq. (51)}$$

(b) Center of mass movement of the whole system

According to Eq. (48), we have

$$\begin{aligned}
x_{\text{c.m.}(s\text{ys.})}(\alpha) &= \frac{\sum_{m,n} x_{\text{c.m.}(m,n)}}{N^2} \\
&= \frac{\sum_{m,n} \left[(2m+n)l \cos \alpha + \frac{l}{2} \cos \alpha - \frac{\sqrt{3}l}{6} \sin \alpha \right]}{N^2} \\
&= \left(\frac{3N-2}{2} \right) l \cos \alpha - \frac{\sqrt{3}l}{6} \sin \alpha
\end{aligned}$$

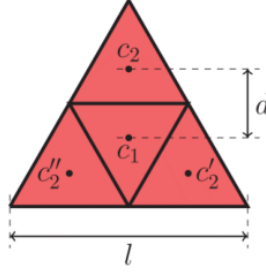
and

$$\begin{aligned}
y_{c.m.(m,n)}(\alpha) &= \frac{\sum_{m,n} y_{c.m.(m,n)}}{N^2} \\
&= - \frac{\sum_{m,n} \left[\sqrt{3}nl \cos \alpha + \frac{\sqrt{3}l}{6} \cos \alpha + \frac{l}{2} \sin \alpha \right]}{N^2} \\
&= - \left(\frac{3N-2}{6} \right) \sqrt{3}l \cos \alpha - \frac{l \sin \alpha}{2} \quad \text{- Eq. (52)}
\end{aligned}$$

Eq. (52) is the trajectory of the center of mass for the whole system, which is not a straight line.

N/A

Appendix 2: Calculation of the moment of inertia of a triangular plate



An equilateral triangle with lateral length l can be divided into four small equilateral triangles with lateral length $l/2$. For the central small triangle centered at c_1 , its moment of inertia is

$$I_1 = \beta \frac{M}{4} \left(\frac{l}{2} \right)^2 \quad \text{- Eq. (53)}$$

For the non-central small triangle centered at c_2 , c_2' and c_2'' ,

$$I_2 = I_1 + \frac{M}{4} d^2 \quad \text{- Eq. (54)}$$

where $d = \sqrt{3}l/6$ is the distance between the centers of triangles 1 and 2. The second term is from the parallel-axis theorem. The moment of inertia of the whole triangle is the sum of the moment of inertia of the four sub-triangles:

$$\beta M l^2 = 4 \times \beta \frac{M}{4} \left(\frac{l}{2} \right)^2 + 3 \times \frac{M}{4} d^2 \quad \text{- Eq.(55)}$$

Thus

$$\beta = \frac{1}{12} \quad \text{- Eq. (56)}$$

N/A

Appendix 3: The minimum force corresponds to the maximum displacement of the exerting point of this force.

Consider the position of vertices A, B, C of a triangle (m,n) :

$$\begin{aligned}
x_{A(m,n)} &= (2m + n) \cos \alpha_m l \\
y_{A(m,n)} &= -\sqrt{3}n \cos \alpha_m l \\
x_{B(m,n)} &= (2m + n + 1) \cos \alpha_m l \\
y_{B(m,n)} &= - \left(\sqrt{3}n \cos \alpha_m + \sin \alpha_m \right) l \\
x_{C(m,n)} &= \left[(2m + n) \cos \alpha_m + \cos \left(\frac{\pi}{3} + \alpha_m \right) \right] l \\
y_{C(m,n)} &= - \left[\sqrt{3}n \cos \alpha_m + \sin \left(\frac{\pi}{3} + \alpha_m \right) \right] l \quad \text{- Eq. (57)}
\end{aligned}$$

Taking derivatives on α on the above coordinates we get

$$\begin{aligned}
\Delta x_{A(m,n)} &= -(2m+n) \sin \alpha_m l \Delta \alpha = -\frac{(2m+n)\sqrt{3}}{2} l \Delta \alpha \\
\Delta y_{A(m,n)} &= \sqrt{3}n \sin \alpha_m (l \Delta \alpha) = \frac{3n}{2} l \Delta \alpha \\
\Delta x_{B(m,n)} &= -(2m+n+1) \sin \alpha_m l \Delta \alpha = -\frac{(2m+n+1)\sqrt{3}}{2} l \Delta \alpha \\
\Delta y_{B(m,n)} &= -\left(-\sqrt{3}n \sin \alpha_m + \cos \alpha_m\right) l \Delta \alpha = \frac{3n-1}{2} l \Delta \alpha \\
\Delta x_{C(m,n)} &= \left[-(2m+n) \sin \alpha_m - \sin \left(\frac{\pi}{3} + \alpha_m\right)\right] l \Delta \alpha = -\frac{(2m+n+1)\sqrt{3}}{2} l \Delta \alpha \\
\Delta y_{C(m,n)} &= -\left[-\sqrt{3}n \sin \alpha_m + \cos \left(\frac{\pi}{3} + \alpha_m\right)\right] l \Delta \alpha = \frac{(3n+1)}{2} l \Delta \alpha \quad - \text{Eq. (58)}
\end{aligned}$$

For $\Delta r = \sqrt{(\Delta x)^2 + (\Delta y)^2}$, we have

$$\begin{aligned}
\Delta r_{A(m,n)} &= \sqrt{3m^2 + 3n^2 + 3mn} (l \Delta \alpha) \\
\Delta r_{B(m,n)} &= \sqrt{3m^2 + 3n^2 + 3mn + 3m + 1} (l \Delta \alpha) \\
\Delta r_{C(m,n)} &= \sqrt{3m^2 + 3n^2 + 3mn + 3m + 3n + 1} (l \Delta \alpha) \quad - \text{Eq. (59)}
\end{aligned}$$

Thus we find

$$\Delta r_{C(m,n)} > \Delta r_{B(m,n)} > \Delta r_{A(m,n)} \quad - \text{Eq. (60)}$$

Therefore, we should choose point C of the triangle $(N-1, N-1)$ to obtain

$$\Delta r_{\max} = (3N-2)l\Delta\alpha \quad - \text{Eq. (61)}$$

so that the force is minimal.