Marking Scheme – T3

Part	**Model Answer**	**(Full mark = 20)**	**Marks**

A1 **2**

The angular momentum should be

$\vec{L} = r\vec{e}_r \times \vec{p} = r\vec{e}_r \times \vec{e}_\varphi \int_0^{2\pi} \frac{mv}{2\pi r} r\, d\varphi$ (1 point for the definition of angular momentum)

Here $\vec{e}_r$ is the unit vector pointing from the center of the ring to the mass point on the ring and $\vec{e}_\varphi$ is the unit vector parallel to the direction of the linear velocity at the mass point.

We know that $v = \omega r$, so finally we can get

$\vec{L} = m\omega r^2 \vec{e}_z$, with $\vec{e}_z = \vec{e}_r \times \vec{e}_\varphi$. (1 point for the correct answer: 0.5 points for the magnitude and 0.5 points for the direction)

A2 **2**

For a current loop, the magnetic moment is defined as
$\vec{M} = I\vec{\mathrm{A}}$

The current can be expressed as

$I = -ef = -e\frac{\omega}{2\pi}$ (1 point for the current expression)

Finally

$$\begin{aligned} \vec{M} &= -e\frac{\omega}{2\pi}\pi r^2 \vec{e}_z \\ &= -\frac{e\vec{L}}{2m} \end{aligned}$$ (1 point for the answer)

A3 **2**

For a current loop, under a uniform magnetic field the total torque should be

$\vec{\tau} = \vec{M} \times \vec{B}$ (0.5 point for the torque definition)

The work done by the magnetic field on the torque should be

$$\begin{aligned} W &= \int_{\frac{\pi}{2}}^{\theta} \vec{\tau} \cdot \mathrm{d}\vec{\theta'} \\ &= \int_{\frac{\pi}{2}}^{\theta} -|\vec{\tau}|\, \mathrm{d}\theta' \\ &= \int_{\theta}^{\frac{\pi}{2}} |\vec{M}||\vec{B}| \sin\theta' \mathrm{d}\theta' \\ &= \vec{M} \cdot \vec{B} \end{aligned}$$ (1.5 points for the work on the torque)

$$\begin{aligned} U &= -W \\ &= -\vec{M} \cdot \vec{B} \\ &= \frac{1}{2} e\omega r^2 B_z \cos\theta \end{aligned}$$ (0.5 point for the answer)

A4 **1**

We assume that the magnetic field is along z direction such that $\vec{B} = B\vec{e}_z$, then in general

$U = -\vec{M} \cdot \vec{B} = -M_z B$

The magnetic torque of an electron should be

$M_z = \frac{-e}{2m_\mathrm{e}} S_z$ (0.5 points for the electron torque)

Thus

$$\begin{aligned} U &= -\vec{M} \cdot \vec{B} \\ &= -\frac{-e}{2m_\mathrm{e}} S_z B \\ &= \frac{\mu_\mathrm{B}}{\hbar} S_z B \\ &= \frac{1}{2}\mu_\mathrm{B} B \end{aligned} \qquad \text{(0.5 points for the answer)}$$

Here $\mu_\mathrm{B} = \frac{e\hbar}{2m_\mathrm{e}}$ is the Bohr magneton.

$\mu_\mathrm{B} = 5.788 \times 10^{-5}\,\mathrm{eV} \cdot \mathrm{T}^{-1}$

A5 **1**

Thus for spin parallel state $S_z = \frac{1}{2}\hbar$, we have

$U = 5.788 \times 10^{-5}\mathrm{eV}$ (0.5 points)

For spin anti-parallel state $S_z = -\frac{1}{2}\hbar$, we have

$U = -5.788 \times 10^{-5}\mathrm{eV}$ (0.5 points)

B1 **1**

In the superconductivity state, electrons forming a Cooper pair have opposite spins, thus the external magnetic field cannot have any effect on the cooper pair. Thus the energy of the Cooper pair does not change.

$E_S = \frac{p_1^2}{2m} + \frac{p_2^2}{2m} - 2\Delta$ (1 point for the answer)

B2 **1**

In the normal state, the two electrons will align their magnetic moments parallel to the external magnetic field. Therefore we have

$$E_\mathrm{N} = \frac{p_1^2}{2m} + \frac{p_2^2}{2m} + \frac{2\mu_\mathrm{B} S_{1x} B_x}{\hbar} + \frac{2\mu_\mathrm{B} S_{2x} B_x}{\hbar}$$

Here the potential energy of electrons should be twice as the classical estimation according to quantum mechanics. Because $S_{1x} = S_{2x} = -\frac{1}{2}\hbar$ can make the magnetic moment aligned along x direction, eventually we have

$$\begin{aligned} E_N &= \frac{p_1^2}{2m} + \frac{p_2^2}{2m} - 2\mu_\mathrm{B} B_x \\ &= \frac{p_1^2}{2m} + \frac{p_2^2}{2m} - \frac{e\hbar}{m_\mathrm{e}} B_x \end{aligned} \qquad \text{(1 point)}$$

B3 **1**

$E_N < E_S \Rightarrow 2B_x\mu_B > 2\Delta \Rightarrow B_x > \frac{\Delta}{\mu_B}$

Thus $B_\mathrm{P} = \frac{\Delta}{\mu_\mathrm{B}} = \frac{2m_\mathrm{e}\Delta}{e\hbar}$ (1 points)

Note: The above simple consideration for the upper critical field B_P over estimates its value. The strict derivation considering the Pauli magnetization and superconductivity condensation energy will give $B_{\mathrm{P}} = \dfrac{\Delta}{\sqrt{2}\mu_{\mathrm{B}}} = \sqrt{2}\,\dfrac{m_{\mathrm{e}}\Delta}{e\hbar}$.

C1 **3**

Method 1:

Substituting $\psi(x) = \left(\dfrac{2\lambda}{\pi}\right)^{\frac{1}{4}} \mathrm{e}^{-\lambda x^2}$ into the $F(\psi)$, we have

$$F(\psi) = \sqrt{\frac{2\lambda}{\pi}} \int_{-\infty}^{+\infty} \mathrm{e}^{-\lambda x^2} \left[-\alpha \mathrm{e}^{-\lambda x^2} - \frac{\hbar}{4m_{\mathrm{e}}} \left(-2\lambda e^{-\lambda x^2} + 4\lambda^2 x^2 \mathrm{e}^{-\lambda x^2} \right) + \frac{e^2 B_z^2 x^2}{m_{\mathrm{e}}} \mathrm{e}^{-\lambda x^2} \right] \mathrm{d}x$$

$$= \sqrt{\frac{2\lambda}{\pi}} \int_{-\infty}^{+\infty} \left[\left(-\alpha + \frac{\hbar^2 \lambda}{2m_{\mathrm{e}}} \right) \mathrm{e}^{-2\lambda x^2} + \left(\frac{e^2 B_z^2}{m_e} - \frac{\hbar^2 \lambda^2}{m_e} \right) x^2 \mathrm{e}^{-2\lambda x^2} \right] \mathrm{d}x$$

$$= -\alpha + \frac{\hbar^2 \lambda}{2m_{\mathrm{e}}} + \left(\frac{e^2 B_z^2}{m_{\mathrm{e}}} - \frac{\hbar^2 \lambda^2}{m_{\mathrm{e}}} \right) \cdot \frac{1}{4\lambda}$$

$$= -\alpha + \frac{\hbar^2 \lambda}{4m_{\mathrm{e}}} + \frac{e^2 B_z^2}{4\lambda m_{\mathrm{e}}}$$

(1.5 points for the correct expression of $F(\psi)$as a function of λ)

We can treat $F(\psi)$ as a function of λ. Thus we have

$$F(\psi) = -\alpha + \frac{\hbar^2 \lambda}{4m_{\mathrm{e}}} + \frac{e^2 B_z^2}{4\lambda m_{\mathrm{e}}}, \text{ and } \frac{\mathrm{d}F}{\mathrm{d}\lambda} = \frac{\hbar^2}{4m_{\mathrm{e}}} - \frac{e^2 B_z^2}{4m_{\mathrm{e}}\lambda^2}.$$

$F(\psi)$ takes the minimum value when $\dfrac{\mathrm{d}F}{\mathrm{d}\lambda} = 0$ and $\dfrac{\mathrm{d}^2 F}{\mathrm{d}\lambda^2} > 0$, thus

$$\frac{\hbar^2}{4m_{\mathrm{e}}} - \frac{e^2 B_z^2}{4m_{\mathrm{e}}\lambda^2} = 0 \qquad \text{(0.5 point for the way to minimize } F(\psi))$$

Finally, we can get

$$\lambda = \frac{eB_z}{\hbar} \qquad \text{(1 point for the correct answer)}$$

We can check that $\dfrac{d^2 F}{d\lambda^2} > 0$ when $\lambda = \dfrac{eB_z}{\hbar}$, which guarantees that $F(\psi)$ takes the minimum value when $\lambda = \dfrac{eB_z}{\hbar}$.

Method 2:

$$F(\psi) = \int_{-\infty}^{+\infty} \psi \left(-\alpha\psi - \frac{\hbar^2}{4m_{\mathrm{e}}} \frac{\mathrm{d}^2\psi}{\mathrm{d}x^2} + \frac{e^2 B_z^2 x^2}{m_{\mathrm{e}}} \psi \right) \mathrm{d}x$$

$$= \int_{-\infty}^{+\infty} \psi \left(-\alpha - \frac{\hbar^2}{4m_{\mathrm{e}}} \frac{\mathrm{d}^2}{\mathrm{d}x^2} + \frac{e^2 B_z^2 x^2}{m_{\mathrm{e}}} \right) \psi \mathrm{d}x \qquad \text{(1 point)}$$

$$= \int_{-\infty}^{+\infty} \psi \tilde{H} \psi \mathrm{d}x$$

In this way, for normalized wave function ψ the $F(\psi)$ is simply the energy expectation $\left\langle \tilde{H} \right\rangle$, the eigenvalue of the Hamiltonian

$$\tilde{H} = -\frac{\hbar^2}{4m_{\mathrm{e}}} \frac{\mathrm{d}^2}{\mathrm{d}x^2} + \frac{e^2 B_z^2}{m_{\mathrm{e}}} x^2 - \alpha$$

The first two terms correspond to the quantum simple harmonic oscillator Hamiltonian. Thus the ground state energy should be

$F_{\min} = \frac{1}{2}\hbar\omega - \alpha$

Here $\omega = \frac{eB_z}{m_\mathrm{e}}$ and ground state wave function becomes

$$\begin{aligned} \psi &= \left(\frac{2m_\mathrm{e}\omega}{\pi\hbar}\right)^{\frac{1}{4}} e^{-\frac{m_\mathrm{e}\omega}{\hbar}x^2} \\ &= \left(\frac{2eB_z}{\pi\hbar}\right)^{\frac{1}{4}} e^{-\frac{eB_z}{\hbar}x^2} \end{aligned} \qquad \text{(1 point)}$$

Therefore, we have

$\lambda = \frac{eB_z}{\hbar}$. (1 point)

C2 **2**

From Part (C1) we know $F_{\min}(\psi) = \frac{\hbar eB_z}{2m_\mathrm{e}} - \alpha$. At the critical value for B_z , it makes the energy difference zero. It means that the critical value B_z satisfies

$\frac{\hbar eB_z}{2m_\mathrm{e}} - \alpha = 0$. (1 point for this equation)

Consequently,

$B_z = \frac{2m_e\alpha}{e\hbar}$. (1 point for the correct answer)

D1 **1**

$E_\mathrm{I} = \frac{p_1^2}{2m} + \frac{p_2^2}{2m} - 2\Delta - \frac{2\mu_\mathrm{B}S_{1z}B_z}{\hbar} + \frac{2\mu_\mathrm{B}S_{2z}B_z}{\hbar}$

Here $S_{1z} = \frac{1}{2}\hbar$, $S_{2z} = -\frac{1}{2}\hbar$

$$\begin{aligned} E_\mathrm{I} &= \frac{p_1^2}{2m} + \frac{p_2^2}{2m} - 2\Delta - \frac{2\mu_\mathrm{B}S_{1z}B_z}{\hbar} + \frac{2\mu_\mathrm{B}S_{2z}B_z}{\hbar} \\ &= \frac{p_1^2}{2m} + \frac{p_2^2}{2m} - 2\Delta - \frac{2\mu_\mathrm{B}B_z}{2} - \frac{2\mu_\mathrm{B}B_z}{2} \\ &= \frac{p_1^2}{2m} + \frac{p_2^2}{2m} - 2\Delta - 2\mu_\mathrm{B}B_z \\ &= \frac{p_1^2}{2m} + \frac{p_2^2}{2m} - 2\Delta - \frac{e\hbar}{m_\mathrm{e}}B_z \end{aligned} \qquad \text{(1 point)}$$

D2 **2**

In the normal state, the electrons will align the magnetic moment parallel to the total magnetic field, thus

$E_{||} = \frac{p_1^2}{2m} + \frac{p_2^2}{2m} + \frac{2\mu_\mathrm{B}\vec{S}_1\cdot\vec{B}_1}{\hbar} + \frac{2\mu_\mathrm{B}\vec{S}_2\cdot\vec{B}_2}{\hbar}$

For electron 1, $\vec{B}_1 = (B_x, 0, -B_z)$

For electron 2, $\vec{B}_2 = (B_x, 0, B_z)$

Therefore, $\vec{S}_1 = -\frac{1}{2}\hbar\left(\frac{B_x}{\sqrt{B_x^2+B_z^2}}, 0, \frac{-B_z}{\sqrt{B_x^2+B_z^2}}\right)$ and $\vec{S}_2 = -\frac{1}{2}\hbar\left(\frac{B_x}{\sqrt{B_x^2+B_z^2}}, 0, \frac{B_z}{\sqrt{B_x^2+}}\right.$ can make the their magnetic moments parallel to the total magnetic field respectively.

(1 point for the correct expression of spins: 0.5 points for each respectively)

Finally

$E_{||} = \frac{p_1^2}{2m} + \frac{p_2^2}{2m} - 2\mu_\mathrm{B}\sqrt{B_x^2+B_z^2} = \frac{p_1^2}{2m} + \frac{p_2^2}{2m} - \frac{e\hbar}{m_e}\sqrt{B_x^2+B_z^2}$ (1 point for the answer)

D3 **1**

$$E_{||} < E_{\text{Ising}} \Rightarrow 2\mu_{\text{B}}\sqrt{B_x^2 + B_z^2} > 2\Delta + 2\mu_{\text{B}}B_z \Rightarrow B_x > \frac{\sqrt{\Delta^2 + 2\Delta\mu_{\text{B}}B_z}}{\mu_{\text{B}}} \quad \text{(1 points)}$$

Another correct expression is: $B_I > \dfrac{2m_e\sqrt{\Delta^2 + \frac{e\hbar}{m_e}\Delta B_z}}{e\hbar}$.