

Vortices in Superfluid

MODD-Problems

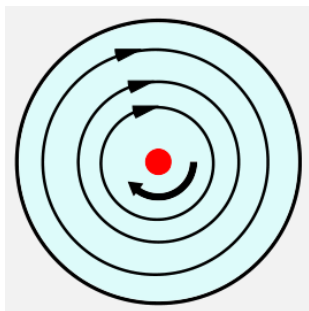
May 5, 2017

A. Steady filament (0.75)

Consider a cylindrical beaker (radius $R_0 \gg a$) of superfluid helium and a straight vertical vortex filament in its center Fig. 2.

A1 (0.25)

Plot the streamlines. Find out the velocity v at a point $\vec{r}$.



The streamlines are circular. From the circulation identity (1) it is obvious that $v = \kappa/r$.

- Streamlines are plotted correctly (one at least)0.1
- $v = \frac{\kappa}{r}$ 0.15

A2 (0.5)

Work out the free surface shape (height as a function of coordinate $z(\vec{r})$) around the vortex. Free fall acceleration is g . Surface tension can be neglected.

Consider a thin circular layer of the radius r . Equilibrium condition for its surface is given by the requirement

$$g \frac{dz}{dr} = \frac{v^2}{r} = \frac{\kappa^2}{r^3}. \quad (1)$$

This equation is satisfied by the surface profile

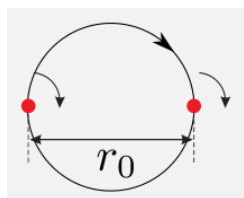
$$z(r) = [z_0] - \frac{\kappa^2}{2gr^2}. \quad (2)$$

- $\tan \alpha = \frac{\kappa^2}{gr^3}$ or equivalent 0.25
- $z = [z_0] - \frac{\kappa^2}{2gr^2}$ 0.25

B. Vortex motion (1.4)

B1 (0.25)

Consider two identical straight vortices initially placed at distance r_0 from each other as shown in Fig. 4. Find initial velocities of the vortices and draw their trajectories.

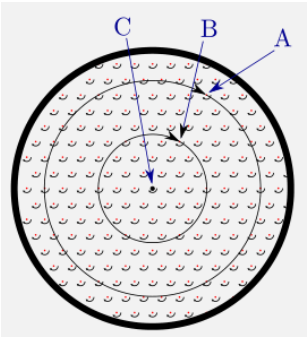


Being advected by each other's flow field, filaments will rotate around a point halfway between them. The velocity is given by $v_0 = \kappa/r_0$.

- Trajectories are plotted correctly 0.15
- Correct expression for velocity0.1

B2 (0.15)

Draw the trajectories of vortices A, B, and C (located in the center).



- Trajectories are plotted correctly 0.15

B3 (0.4)

Find velocity $v(\vec{r})$ of a vortex positioned at $\vec{r}$.

Consider a circular path of radius $r \gg u$ around the beaker center. The circulation along this path is given by the number of vortices within it (vortex density per unit area is $(u^2\sqrt{3}/2)^{-1}$):

$$2\pi r v = 2\pi \kappa \frac{\pi r^2}{u^2\sqrt{3}/2}. \tag{3}$$

The velocity field

$$v = \frac{2\pi \kappa r}{u^2\sqrt{3}}. \tag{4}$$

- Expression for vortex density 0.2
- Correct expression for $v(r)$ 0.2

B4 (0.35)

Find the distance $AB(t)$ between the vortices A and B at time t . Treat $AB(0)$ as given.

This velocity pattern corresponds to the rotation of the lattice as a whole around the beaker center with angular velocity

$$\omega = \frac{2\pi \kappa}{u^2\sqrt{3}}. \tag{5}$$

$$AB(t) = AB(0)$$

- Correct answer 0.35

B5 (0.25)

Work out the “smoothed out” (omitting the lattice structure) free helium surface shape $z(\vec{r})$.

The surface shape is

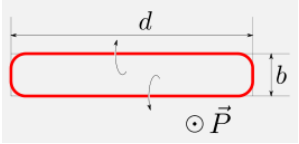
$$z(r) = [z_0] + \frac{\omega^2 r^2}{2g} = [z_0] + \frac{2\pi^2 \kappa^2 r^2}{3gu^4}. \tag{6}$$

- Correct answer 0.25

C. Momentum and Energy (1.75)

C1 (0.3)

Consider a nearly rectangular vortex loop $b \times d$, $b \ll d$, Fig. 7. Indicate the direction of its momentum $\vec{P}$. Find out the momentum magnitude.



Momentum of a flat loop (see Introduction) is perpendicular to its plane and proportional to its area. For a rectangular loop the magnitude is $P = 2\pi\kappa\rho bd$.

- Correct direction of momentum0.15
- Correct expression for momentum magnitude 0.15

C2 (0.7)

Calculate its energy U .

To produce equal magnetic and kinetic energy densities $B^2/(2\mu_0) = \rho v^2/2$, the magnetic field has to be $B = v\sqrt{\mu_0\rho} = \kappa\sqrt{\mu_0\rho}/r$. This field is generated by a current $I = 2\pi\kappa\sqrt{\rho/\mu_0}$. Energy of the wire loop can be found from the inductance $U = LI^2/2$. Inductance of a nearly rectangular wire loop:

$$L = \frac{\Phi}{I} = 2dI^{-1} \int_a^b \frac{\mu_0 I}{2\pi r} dr = \frac{\mu_0 d}{\pi} \log \frac{b}{a}. \tag{7}$$

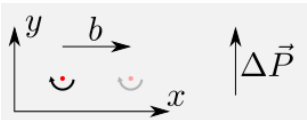
This gives for the energy

$$U = 2\pi\kappa^2\rho d \log \frac{b}{a} \tag{8}$$

- Integration limits are $\sim a$ and b 0.2
- Analogy with a magnitude field is used ($U = \frac{LI^2}{2}$, $L = \frac{\Phi}{I}$) or energy is calculated as $W = \int F dr$, where $F = \frac{dP}{dt}$ 0.2
- Correct expression for energy0.3

C3 (0.75)

Suppose we shift a long straight vortex filament by a distance b in x direction, see Fig. 8. How much does the fluid momentum change? Indicate the momentum change direction. The filament length (constrained by the vessel walls) is d .



The momentum change is equal to the momentum of a long rectangular loop $P = 2\pi\kappa\rho bd$.

- The result of C1 used0.3
- Momentum change is parallel to Y axis 0.1

- Correct direction of momentum change 0.15
- Correct expression for momentum change magnitude 0.2

Interestingly, this provides an alternative approach to find the energy of such a loop. Namely, if we slowly move one straight vortex in the velocity field of another, then we apply a force

$$F = 2\pi\kappa\rho dv = 2\pi\kappa\rho d\frac{\kappa}{r} = \frac{2\pi\kappa^2\rho d}{r}.$$
(9)

The work

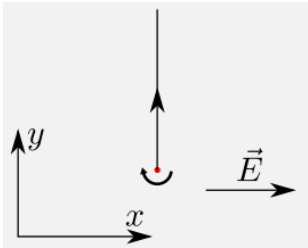
$$W = \int_a^b \frac{2\pi\kappa^2\rho d}{r}dr = 2\pi\kappa^2\rho d\log\frac{b}{a}$$
(10)

has to be performed to move it from distance a to b .

D. Trapped charges (2.85)

D1 (0.5)

Consider a straight vortex charged with uniform linear density $\lambda < 0$ in a uniform electric field $\vec{E}$. Draw the vortex trajectory. Find its velocity as a function of time.



Electric force $F = E\lambda d$ moves the vortex with velocity

$$v = \frac{F}{2\pi\kappa\rho d} = \frac{E\lambda}{2\pi\kappa\rho}$$
(11)

perpendicular to $\vec{E}$.

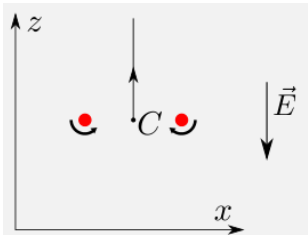
- Trajectory is straight line parallel to Y axis 0.1
- Correct direction of velocity 0.2
- Correct expression for velocity magnitude 0.2

D234

A circular vortex loop of radius R_0 initially charged with uniform linear density $\lambda < 0$ is placed in a uniform electric field $\vec{E}$ perpendicular to its plane, opposite to its momentum $\vec{P}_0$.

D2 (0.6)

Draw the trajectory of the loop center C . Find the radius of the loop as a function of time.



Electric force upon the loop $F = -2\pi ER_0|\lambda|$ is constant and fluid momentum linearly depends on time

$$P = P_0 + 2\pi ER_0|\lambda|t = 2\pi^2\rho R^2\kappa. \tag{12}$$

The loop is growing and its radius is increasing with time t

$$R = \sqrt{R_0^2 + \frac{ER_0|\lambda|t}{\pi\rho\kappa}}. \tag{13}$$

- Trajectory is straight line along y 0.1
- Correct velocity direction0.15
- $P(t)$ 0.15
- $2\pi^2\rho R^2\kappa$ 0.15
- Correct expression for $R(t)$ 0.05

D3 (1.5)

Find its velocity $v(t)$ as a function of time.

The loop velocity v can be easily found from a relationship between the energy change rate and the momentum change rate

$$\frac{dU}{dt} = Fv = \frac{dP}{dt}v. \tag{14}$$

This gives for the velocity

$$v = \frac{dU}{dP} \approx \frac{\kappa}{2R} \log \frac{R}{a} = \frac{\kappa \log \left(\sqrt{R_0^2 + ER_0|\lambda|t/(\pi\rho\kappa)}/a \right)}{2\sqrt{R_0^2 + ER_0|\lambda|t/(\pi\rho\kappa)}} \approx \frac{\kappa \log(R_0/a)}{2\sqrt{R_0^2 + ER_0|\lambda|t/(\pi\rho\kappa)}}. \tag{15}$$

This means that the vortex is moving in the direction of the force but its velocity is decreasing.

- Expression for $v \propto \frac{1}{R} \log(R)$ 1.0
- Correct expression for $v(t)$ 0.5

D4 (0.25)

The field is switched off at a time t^* when the velocity reaches the value $v^* = v(t^*)$. Find the loop velocity $v(t)$ at a later time $t > t^*$.

When $E = 0 \Rightarrow P = \text{const} \Rightarrow R = \text{const} \Rightarrow v = \text{const} \Rightarrow v(t) = v^*$.

- Correct expression for $v(t)$ 0.25

E. Influence of the boundaries (3.25)

Draw the trajectory of a straight vortex, initially placed at a distance h_0 from a flat wall. Find its velocity as a function of time.

E1 (0.5)

Well known technique of image charges (currents) in electrostatics (magnetostatics) can be directly used to solve this problem. Namely, the wall can be “substituted” with a reflected fictitious vortex on the other side of the wall. The velocity distribution of two vortices together in the upper semi-space is identical to the one produce by a single

vortex above the wall. Indeed, the symmetry of the problem ensures that there is no flow through the plane of symmetry. Thus, a straight vortex line situated a distance h_0 above a flat wall with its image behave as a pair of vortices of opposite circulation a distance $2h_0$ apart. This means that the vortex moves along the wall with velocity

$$v = \frac{\kappa}{2h_0}.$$

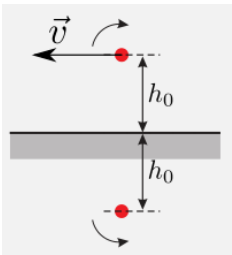


Illustration of the image method for the straight vortex filament near a flat wall

- Trajectory is plotted correctly 0.15
- Correct direction of velocity 0.1
- Correct expression for velocity magnitude0.25

E234

Consider a straight vortex placed in a corner at a distance h_0 from both walls.

E2 (0.75)

What is the initial velocity v_0 of the vortex?

The velocity of the filament is given by superposition of the velocities $\vec{v}_1$, $\vec{v}_2$ and $\vec{v}_3$ induced by the image vortices 1, 2 and 3, respectively (see Fig. in E3 solution). One readily obtains

$$v_1 = \frac{\kappa}{2h_0}, \quad v_2 = \frac{\kappa}{2\sqrt{2}h_0}, \quad v_3 = \frac{\kappa}{2h_0}.$$

The modulus of the filament velocity at the initial moment is

$$v_0 = |\vec{v}_1 + \vec{v}_2 + \vec{v}_3| = \sqrt{2}v_1 - v_2 = \boxed{\frac{\kappa}{2\sqrt{2}h_0}}$$

- Ideas of using superposition principal and technique of image charges 0.25
- Correct expression for initial velocity magnitude 0.5

E3 (0.5)

Draw the trajectory of the vortex.

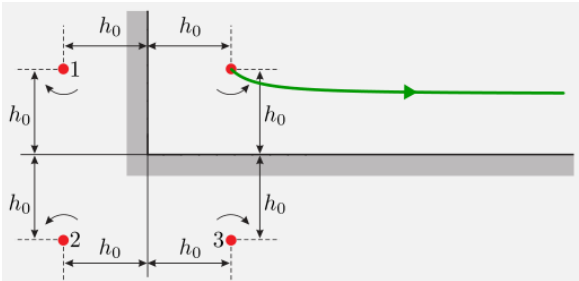


Image vortices in the corner.

- The trajectory has correct form0.3
- Correct direction of initial velocity0.2

E4 (1.5)

What is the velocity of the vortex v_∞ after very long time?

Energy for the system of vortices is proportional to

$$U_{\text{tot}} \propto \log \frac{\sqrt{x^2 + y^2}}{a} - \log \frac{x}{a} - \log \frac{y}{a}. \quad (17)$$

The energy conservation implies that

$$C = \frac{x^2 + y^2}{x^2 y^2} = \frac{2}{h_0^2} \quad (18)$$

is constant along the trajectory. After very long time $y \rightarrow h_0/\sqrt{2}$ and the vortex velocity is

$$v_\infty = \frac{\kappa}{h_0 \sqrt{2}}. \quad (19)$$

- $E = \text{const}$ 0.5
- Correct expression for velocity after very long time 1.5