Theory Q1
Optical trap of neutral atoms (12 points),
Solution and Marking Scheme

1.1 0.75pt

At the instance when the seperation between charge centers is $\vec{x}$, the external field $\vec{E}$ exerts on them opposite forces $\vec{F} = \pm e\vec{E}$. **0.15**

After a time interval dt, the seperation is changed to $\vec{x}+d\vec{x}$, work done by the external field on the charges is thus $dW = \vec{F}d\vec{x} = \vec{F} = ed\vec{x}\cdot\vec{E} = d\vec{p}\cdot\vec{E}$ **0.3**

The power received by the atomic dipole

$$P_{abs} = \frac{dW}{dt} = \frac{d\vec{p}}{dt}\cdot\vec{E} = \dot{\vec{p}}\cdot\vec{E} \qquad \textbf{0.3}$$

1.2 0.75pt

Total work can be obtained by integration **0. 5**

$$W = \int_0^{\vec{E}_0} d\vec{p}\cdot\vec{E} = \int_0^{\vec{E}_0} \alpha d\vec{E}\cdot\vec{E} = \frac{1}{2}\alpha\vec{E}_0^2 = \frac{1}{2}\vec{p}_0\vec{E}_0$$

Potential energy of the dipole is

$$U_{dip} = -W = -\frac{1}{2}\vec{p}_0\vec{E}_0 \qquad \textbf{0.25}$$

If the sign of U_{dip} is incorrect or the factor 1/2 is missing, students get 0pt.

2.1 1.0pt

The time average of any time dependent function is denoted by $\langle f(t)\rangle = \frac{\omega}{2\pi}\int_0^{2\pi/\omega} f(t)\,dt$

$$U_{dip}(\vec{r}) = -\frac{1}{4}\alpha(\omega)\cos\varphi.E_0^2(\vec{r}) \qquad (1) \qquad \textbf{0.5}$$

$$U_{dip}(\vec{r}) = -\frac{\alpha(\omega)\cos\varphi.I(\vec{r})}{2\varepsilon_0 c} \qquad (2) \qquad \textbf{0.5}$$

If student gets directly to eq. (2) – full mark (1.0pt)

If the answer is still correct but expressed in any quantity other than those requested – 0.5 pt.

3.1 1.0pt

The power absorbed by the oscillator from the driving field (and re-emitted as dipole radiation) is given by

$$\langle P_{abs}(\vec{r})\rangle = \left\langle \dot{\vec{p}}\vec{E}\right\rangle = -\frac{\sin\varphi.\alpha(\omega)\,\omega}{2}E_0^2(r) \qquad \textbf{0.5}$$

$$\langle P_{abs}(\vec{r})\rangle = -\frac{\sin\varphi.\alpha(\omega)\,\omega}{\varepsilon_0 c}I(\vec{r}) \qquad (3) \qquad \textbf{0.25}$$

The corresponding *scattering rate* is $\Gamma_{sc}(\vec{r}) = \frac{\langle P_{abs}\rangle}{\hbar\omega} = -\frac{\alpha(\omega)\sin\varphi}{\hbar\varepsilon_0 c}I(\vec{r}).(4)$ **0.25**

4.1 2.0pt

In one dimensional Lorentz's model, we replace $\vec{E}(\vec{r},t) \to E(x,t)$. One can find the solution of the form $x = x_0 \cos(\omega t + \varphi)$ thus from the equation of motion,

$$\ddot{x} + \gamma_\omega \dot{x} + \omega_0^2 x = -eE_0 \cos\omega t/m_e$$

$$=> x_0\left(\omega_0^2 - \omega^2\right)\cos(\omega t + \varphi) - x_0\omega\gamma_\omega \sin(\omega t + \varphi) = -eE_0 \cos\omega t/m_e \quad \textbf{0.25}$$

$$x_0\left\{\left[\left(\omega_0^2 - \omega^2\right)\cos\varphi - \omega\gamma_\omega \sin\varphi\right]\cos\omega t - \left[\left(\omega_0^2 - \omega^2\right)\sin\varphi + \omega\gamma_\omega \cos\varphi\right]\sin\omega t\right\} = -eE_0 \cos\omega t/m_e \quad \textbf{0.25}$$

Comparing coefficients before $\cos\omega t$ and$\sin\omega t$ on both sides, one has

$$\left(\omega_0^2 - \omega^2\right)\cos\varphi - \omega\gamma_\omega \sin\varphi = -\frac{eE_0}{m_e x_0} \quad \textbf{0.5}$$

$$\left(\omega_0^2 - \omega^2\right)\sin\varphi + \omega\gamma_\omega \cos\varphi = 0$$

$$x_0 = \frac{eE_0/m_e}{\sqrt{\left(\omega_0^2 - \omega^2\right)^2 + \gamma_\omega^2\omega^2}}; \quad \textbf{0.25}$$

$$\sin\varphi = \frac{\omega\gamma_\omega}{\sqrt{\left(\omega_0^2 - \omega^2\right)^2 + \gamma_\omega^2\omega^2}} \quad (4)$$

0.25

$$\cos\varphi = -\frac{(\omega_0^2 - \omega^2)}{\sqrt{\left(\omega_0^2 - \omega^2\right)^2 + \gamma_\omega^2\omega^2}} \quad (5)$$

$$p = -ex = -ex_0\cos(\omega t + \varphi) = \alpha E_0 \cos(\omega t + \varphi) \quad (6) \quad \textbf{0.25}$$

$$\alpha(\omega) = -\frac{e^2}{m_e\sqrt{\left(\omega_0^2 - \omega^2\right)^2 + \gamma_\omega^2\omega^2}} \quad (7) \quad \textbf{0.25}$$

Note: students can obtain φ via any of sin, cos, tan functions: full mark (0.25 pt)

5.1 1.0pt

The power radiated due to the damping force, thus

$$-m_e\gamma_\omega v.v = -\frac{1}{6\pi\varepsilon_0}\frac{e^2a^2}{c^3} \quad \textbf{0.25}$$

$$\Rightarrow -m_e\gamma_\omega(\omega r)^2 = -\frac{1}{6\pi\varepsilon_0}\frac{e^2(\omega^2 r)^2}{c^3}, \quad \textbf{0.25}$$

$$\gamma_\omega = \frac{1}{6\pi\varepsilon_0}\frac{e^2\omega^2}{m_e c^3}. \quad (8) \quad \textbf{0.5}$$

6.1 0.5pt

Substituting $e^2/m_e = 6\pi\varepsilon_0 c^3\gamma_\omega/\omega^2$ the on-resonance damping rate $\gamma \equiv \gamma_{\omega_0} = (\omega_0/\omega)^2\gamma_\omega$.

Using Eq. (1), (4), (5) and (6) one has

$$\frac{U_{dip}(\vec{r})}{\hbar\Gamma_{sc}(\vec{r})} = \frac{-\frac{1}{2\varepsilon_0 c}\alpha(\omega)\cos\varphi}{-\hbar\frac{\alpha(\omega)\sin\varphi}{\hbar\varepsilon_0 c}} = \frac{1}{2}\frac{1}{\tan\varphi} = -\frac{1}{2}\frac{\omega_0^2 - \omega^2}{\left(\omega^3/\omega_0^2\right)\gamma}, \quad \textbf{0.5}$$

7.1 0.5pt

From (1), (5) and (6) one has

$$U_{depth} = |U_0| = \left|\frac{\alpha(\omega)\cos(\varphi) I(0,0)}{2\varepsilon_0 c}\right| = \left|\frac{\alpha(\omega)\cos(\varphi)}{2\varepsilon_0 c}\frac{2P}{\pi D_0^2}\right| = \left|6c^2 \frac{\left(\omega_0^2-\omega^2\right)\gamma/\omega_0^2}{\left[\left(\omega_0^2-\omega^2\right)^2+\gamma^2\frac{\omega^6}{\omega_0^4}\right]}\frac{P}{D}\right|$$

0.5

7.2 1.0pt

Trap depth when $P = 4mW$, laser wavelength $\lambda = 985nm$, and $D_0 = 6\mu m$. For sodium $\lambda_0 = 589nm$.

One has: $\omega = \frac{2\pi c}{\lambda}; \omega_0 = \frac{2\pi c}{\lambda_0};$

And $\gamma = \frac{1}{6\pi\varepsilon_0}\frac{e^2\omega_0^2}{m_e c^3} = \frac{2\pi e^2}{3\varepsilon_0 m_e c\lambda_0^2} = 6.4\times 10^7 s^{-1}$ **0.5**

$U_{depth} = f\, k_B T_0$ *(factors f = 3/2, 1/2,1 are all accepted)* **0.25**

$\Rightarrow f.T_0 = 4.13\mu K$ **0.25**

8.1 0.5pt

Using linear expansion, we have $\Omega_\rho = \sqrt{\frac{4k_B f T_0}{mD_0^2}}$ **0.25**

and $\Omega_z = \sqrt{\frac{2k_B f T_0}{mz_R^2}}$ **0.25**

9.1 0.5pt

Mean potential enery $U(z_0) = const + \frac{1}{2}m\Omega_z^2 z_0^2$.

To estimate the particle momentum, we assume $p \sim \Delta p, \Delta z \sim z_0$. **0.2**

The uncertainty principle is written now $p \sim \frac{\hbar}{z_0}$. **0.1**

Kinetic energy $K = \frac{p^2}{2m} = \frac{\hbar^2}{2mz_0^2}$.

Total energy of the particle $E = \frac{1}{2}m\Omega_z^2 z_0^2 + \frac{\hbar^2}{2mz_0^2} + const$ **0.1**

Minimal energy corresponds to the energy balance $\frac{1}{2}m\Omega_z^2 z_0^2 = \frac{\hbar^2}{2mz_0^2} \Rightarrow z_0 = \sqrt{\frac{\hbar}{m\Omega_z}}$.

0.1

If the student followed a correct analysis any obtained correct answer upto some multiplication factor: full mark

If the student obtained correct answer using dimensional analysis: only 0.1 pt is granted

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9.2 0.25pt

Insert the expression of the cloud size $z_0 = \sqrt{\frac{\hbar}{m\Omega_z}}$ to the energy expression

$E_{\min} = \frac{1}{2}m\Omega_z^2 z_0^2 + \frac{\hbar^2}{2mz_0^2} + const$ one obtains $E_{\min} = \hbar\Omega_z + const.$ **0.25**

If the student obtained the answer $E_{min} = \dfrac{\hbar\Omega_z}{2}$ *by using* $E_n = \hbar\Omega_z\left(n + \dfrac{1}{2}\right)$ *: full mark*

9.3 0.25pt

From the uncertainty principle, the particle velocity therefore is estimated to be **0.25**

$$mv_z = \frac{\hbar}{z_0} = \sqrt{m\hbar\Omega_z} \Rightarrow v_z = \sqrt{\frac{\hbar\Omega_z}{m}}.$$

Alternative estimation is constructed from kinetic energy: $\dfrac{1}{2}mv_z^2 = K = \dfrac{1}{2}\hbar\Omega_z \Rightarrow v_z = \sqrt{\dfrac{\hbar\Omega_z}{m}}$

10.1 0.5pt

For the three dimendional trap, one has: $z_0 = \sqrt{\dfrac{\hbar}{m\Omega_z}}$. **0.2**

Similarly for x, y coordinates $x_0 = y_0 = \sqrt{\dfrac{\hbar}{m\Omega_\rho}}$ and thus $\rho_0 = \sqrt{x_0^2 + y_0^2} = \sqrt{\dfrac{2\hbar}{m\Omega_\rho}}$. **0.2**

The condensate aspect ratio: $\dfrac{z_0}{\rho_0} = \sqrt{\dfrac{\Omega_\rho}{2\Omega_z}}$. **0.1**

Student may use either x_0, y_0 *or* ρ_0 *in estimating the radial size of the cloud. Correct answers upto multiplication factor: full mark*

10.2 0.5pt

$$v_z = \sqrt{\frac{\hbar\Omega_z}{m}},$$

$$v_x = v_y = \sqrt{\frac{\hbar\Omega_\rho}{m}}, \Rightarrow v_\rho = \sqrt{v_x^2 + v_y^2} \sim \sqrt{\frac{2\hbar\Omega_\rho}{m}}, \quad \textbf{0.25}$$

$$\frac{v_\rho}{v_z} \sim \sqrt{\frac{2\Omega_\rho}{\Omega_z}}. \quad \textbf{0.25}$$

Student may use either v_x, v_y *or* v_ρ *in estimating expansion velocity in the radial direction. Correct answers upto some multiplication factor: full mark*

10.3 0.5pt

After the time t, the sizes of the condensate cloud are: **0.25**

$$z_L = z_0 + v_z t \approx v_z t \qquad \rho_L = \rho_0 + v_\rho t \approx v_\rho t.$$

The cloud aspect ratio after the time t, $\dfrac{z_L}{\rho_L} \approx \dfrac{v_z}{v_\rho} \sim \sqrt{\dfrac{\Omega_z}{2\Omega_\rho}} \ll 1$. **0.25**

Correct final answers upto some multiplication factor: full mark

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10.4 0.5pt

Due to isotropic nature of thermal cloud, described by the Maxwell distribution:

$$v_{T,z} = v_{T,\rho} \Rightarrow \frac{v_{T,\rho}}{v_{T,z}} \approx 1. \quad \textbf{0.2}$$

one can easily find $z_{T,L} = z_0 + v_z t \approx v_z t$, $\rho_{T,L} = \rho_0 + v_\rho t \approx v_\rho t$. **0.2**

After a very long time, the aspect ratio of the thermal cloud therefore:

$$\rho_{T,L} : z_{T,L} \sim 1 \qquad \textbf{0.1}$$

Note: students use different velocities (arithmetrical, rms, projection….etc.) to estimate the expansion of the cloud, as long as they give the correct ratio $\rho_L : z_T \sim 1$, full mark of this sub question is granted. In this question, the correct multiplication factor is requested. For incorrect multiplication factor: zero mark