Theory Q2

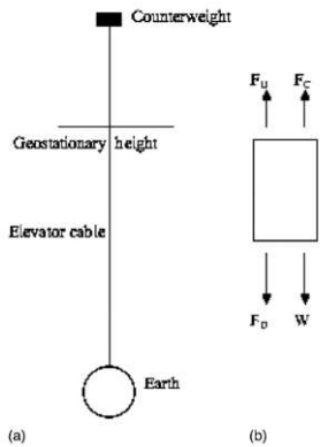
Space elevator (8 points)

Solution and Marking Scheme

1 Cylindrical Space Elevator with Uniform Cross Section

1.1 0.5pt

Consider a small element of the cylinder of thickness dr at position r, there are four forces acting on that element: gravitational $\vec{W}(r)$, centrifugal $\vec{F}_C(r)$, cable tension $\vec{F}_D = \vec{T}(r)$ at position r, tension $\vec{F}_U = \vec{T}(r+dr)$ at position $r+dr$. Positive direction is chosen from the Earth center outward. The net force must be zero, therefore:



$$\begin{aligned} &-W + F_C + T(r+dr) - T(r) = 0 \\ &\Leftrightarrow -W + F_C + A.\sigma(r+dr) - A.\sigma(r) = 0' \end{aligned} \qquad \textbf{0.1}$$

Hence

$$Ad\sigma = \frac{GM(A\,dr\,\rho)}{r^2} - (A\,dr\,\rho)\,\omega^2 r$$

$$\Rightarrow \frac{d\sigma}{dr} = GM\rho\left(\frac{1}{r^2} - \frac{r}{R_G^3}\right) \qquad \textbf{0.1}$$

Note that, the tensions at the ends of the cylinder are zero. Integrating the above equation from R to R_G, one obtains the stress at R_G

$$\sigma(R_G) = GM\rho\left[\frac{1}{R} - \frac{3}{2R_G} + \frac{R^2}{2R_G^3}\right], \qquad \textbf{0.1}$$

Similarly, integrating from R_G to H (the distance from the Earth center to the upper end of the cylinder), one obtains the same stress at R_G

$$\sigma(R_G) = GM\rho\left[\frac{1}{H} - \frac{3}{2R_G} + \frac{H^2}{2R_G^3}\right] \qquad \textbf{0.1}$$

Equating the two above expressions, one arrives to the equation:

$$R\,H^2 + R^2 H - 2R_g^3 = 0,$$

from where H is determined: $H = \dfrac{R}{2}\left[\sqrt{1+8\left(\dfrac{R_G}{R}\right)^3} - 1\right] = 1.51 \times 10^5$ km.

The height of the cylinder $L = H - R = \dfrac{R}{2}\left[\sqrt{1+8\left(\dfrac{R_G}{R}\right)^3} - 3\right] = 1.45 \times 10^5$ km.

0.1

Note: Students can just equalize the net gravitational force and the net centrifugal force acting on the cylinder to obtain H correctly: full mark.

1.2 0.5pt

The maximal stress is determined from the requirement

$$\frac{d\sigma}{dr} = GM\rho\left(\frac{1}{r^2} - \frac{r}{R_G^3}\right) = 0 \qquad \textbf{0.25}$$

which yields $r = R_G$ **0.25**

1.3 0.5pt

Maximal stress is expressed by

$$\sigma\left(R_G\right) = GM\rho\left[\frac{1}{R} - \frac{3}{2R_G} + \frac{R^2}{2R_G^3}\right] \qquad (1)$$

$$\sigma\left(R_G\right) = \rho g\left[R - \frac{3R^2}{2R_G} + \frac{R^4}{2R_G^3}\right] \qquad (2) \qquad \textbf{0.25}$$

Numerical calculation with $\rho = 7900kg/m^3$ one obtains the ratio:

$$\frac{\sigma\left(R_G\right)}{5.0GA} = \frac{383\ GPa}{5.0\ \ GPa} = 76.5, \qquad \textbf{0.25}$$

This ratio is much larger than 1, therefore steel is not suitable to build this kind of elevator.

If eq. (2) is not obtained and other correct equation like eq. (1) is derived - 0.1pt from full mark (get only 0.15pt for maximal stress).

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2 Carbon Nanotubes

2.1 0.25pt

Expand exponential function in series, and limit to the lowest power of x, one has $V = V_0\left(-1 + \frac{4x^2}{a^2}\right)$ and gets $P = -V_0$ and **0.1**

$Q = \frac{4V_0}{a^2}$. **0.15**

2.2 0.25pt

$F = -\frac{dV}{dx} = -\frac{8V_0}{a^2}x$ **0.1**

then $k = \frac{8V_0}{a^2} = 313Nm^{-1}$. **0.15**

2.3 0.5pt

Young's modulus of the carbon nanotube. Denote d the diameter of the carbon nanotube, one has $d = 27b/\pi$.

$E_1 = \frac{stress\ \sigma}{strain\ \varepsilon} = \frac{F/A}{x/a} = \frac{kx/A}{x/a} = \frac{ka}{A} = \frac{32V_0}{a\pi d^2}$ **0.25**

$E = NE_1 = 342\,\mathrm{GPa}$ **0.25**

2.4 0.5pt

$V_0 = \frac{1}{2}kx_{\max}^2 \Rightarrow x_{\max} = \sqrt{\frac{2V_0}{k}} = \frac{1}{2}a$ **0.25**

$= 0.071\,\text{nm}$ **0.25**

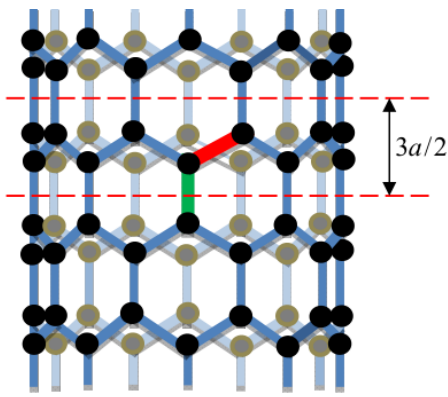
2.5 0.5pt

Tensile strength of the carbon nanotube, $\sigma_0 = E\frac{x_{\max}}{a} = E/2 = 171\text{GPa}$. **0.5**

2.6 0.5pt

Volume $\frac{\pi d^2}{4} \times \frac{3a}{2}$ contains 18 carbon atoms, therefore the density of the carbon nanotube, **0.25**

$$\rho = \frac{2 \times 27 \times 12 \times 10^{-3}}{N_A \times \frac{\pi d^2}{4} \times \frac{3a}{2}} = 1440\,\text{kg/m}^3. \quad \textbf{0.25}$$

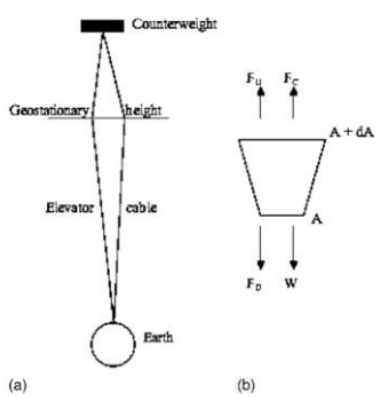


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3 Tapered Space Elevator with Uniform Stress

3.1 0.5pt

The solution to this section is analogous to that given in the previous section, however, now one has to take into account the fact that the stress σ is constant, but the cross section area A varies along the tower.



$$\sigma dA = \frac{GM\,(A\,dr\,\rho)}{r^2} - (A\,dr\,\rho)\,\omega^2 r$$

$$\Rightarrow \frac{dA}{A} = \frac{\rho g R^2}{\sigma}\left(\frac{1}{r^2} - \frac{r}{R_G^3}\right)dr \qquad \textbf{0.25}$$

where $g = GM/R^2$ is gravitational acceleration at the Earth surface. By integration one can obtain the tower cross section as:

$$A(h) = A_S \exp\left[\frac{\rho g R^2}{\sigma}\left(\frac{1}{R} + \frac{R^2}{2R_G^3} - \frac{1}{R+h} - \frac{(R+h)^2}{2R_G^3}\right)\right] \qquad \textbf{0.25}$$

3.2 0.5pt

Using the condition A(H)=A(R)=A_S one arrives to the equation $R\,H^2 + R^2 H - 2R_G^3 = 0$, which allows to determine **0.25**

$$H = \frac{R}{2}\left[\sqrt{1 + 8\left(\frac{R_G}{R}\right)^3} - 1\right] = 151000\,\text{km}. \qquad \textbf{0.25}$$

3.3 0.5pt

The ratio $\frac{A_G}{A_S} = \exp[\frac{R}{2L_C}\{\left(\frac{R}{R_G}\right)^3 - 3\left(\frac{R}{R_G}\right) + 2\}] = 1.623$ where $L_C = \frac{\sigma}{\rho g}$ **0.5**

3.4 1.0pt

Net force exerted on the counterweight must be zero

$$\frac{GMm_C}{[R_G + h_C]^2} + A(R_G + h_C).\sigma = m_C\omega^2[R_G + h_C], \qquad \textbf{0.5}$$

replacing $A(R_G + h)$ from the equation for cross section area, one can determine the counterweight mass.

$$m_C = \frac{\rho A_S L_C \exp\left[\frac{R^2}{2L_C R_G^3}\left(\frac{2R_G^3 + R^3}{R} - \frac{2R_G^3 + (R_G + h_C)^3}{R_G + h_C}\right)\right]}{\frac{R^2(R_G + h_C)}{R_G^3}\left[1 - \left(\frac{R_G}{R_G + h_C}\right)^3\right]}. \qquad \textbf{0.50}$$

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4 Applications

4.1 0.5pt

An object can leave the Earth if its energy at the distance r satisfies

$E = \frac{m(\omega r)^2}{2} - \frac{GMm}{r} \geq 0$ from which $r_C = \left(2GM/\omega^2\right)^{\frac{1}{3}} = 53200km$ **0.25**

In order to launch an object, the upper end of the tower must locate above the distance r_C. **0.25**

4.2 1.0pt

We denote the Earth orbital velocity as v_E, the spacecraft velocity when it's released from the tower top as $v_1 = \omega h_0$. The spacecraft can reach the furthest distance from the Sun if $\vec{v}_1$ is parallel to $\vec{v}_E$. The spacecaft velocity relative to the Sun is $v_E + v_1$. The Earth orbital radius R_E also is the smallest distance from the sun (if one neglects the tower length compared to the radius of the Earth's orbit). r_2 is the apogee distance of the spacecraft from the Sun, v_2 is its velocity at apogee.

Angular momentum and energy convervation laws read

$$m\left(v_E+v_1\right)R_E=mv_2r_2 \qquad \textbf{0.1}$$

$$\frac{1}{2}m\left(v_E+v_1\right)^2-\frac{GM_Sm}{R_E}=\frac{1}{2}mv_2^2-\frac{GM_Sm}{r_2} \qquad \textbf{0.1}$$

Here the energy term $-\dfrac{GMm}{h_0}$ due the earth's gravity is neglected. Eliminating v_2 one has

$$\left[\left(v_E+\omega h_0\right)^2-\frac{2GM_S}{R_E}\right]r_2^2+2GM_Sr_2-\left(v_E+\omega h_0\right)^2R_E^2=0 \qquad \textbf{0.1}$$

$$\text{from which } r_{Max}=r_2=\frac{\left(v_E+\omega h_0\right)^2R_E^2}{2GM_S-\left(v_E+\omega h_0\right)^2R_E}. \qquad \textbf{0.1}$$

Numerical calculation gives r_2=5.3AU, that covers Jupiter's orbit. **0.1**

Similarly, for the spacecraft to approach as close as possible to the Sun, the released velocity $\vec{v}_1$ must be antiparallel to $\vec{v}_E$. The spacecaft velocity relative to the Sun is v_E-v_1, r_2 is the perigee distance of the spacecraft from the Sun, v_2 is its velocity at perigee.

The previous angular momentum and energy convervation laws still hold,

$$m\left(v_E-v_1\right)R_E=mv_2r_2 \qquad \textbf{0.1}$$

$$\frac{1}{2}m\left(v_E-v_1\right)^2-\frac{GM_Sm}{R_E}=\frac{1}{2}mv_2^2-\frac{GM_Sm}{r_2} \qquad \textbf{0.1}$$

Here the energy term $-\dfrac{GMm}{h_0}$ due the earth's gravity is neglected. Eliminating v_2 one has

$$\left[\left(v_E-\omega h_0\right)^2-\frac{2GM_S}{R_E}\right]r_2^2+2GM_Sr_2-\left(v_E-\omega h_0\right)^2R_E^2=0 \qquad \textbf{0.1}$$

$$\text{from which } r_{\min}=r_2=\frac{\left(v_E-\omega h_0\right)^2R_E^2}{2GM_S-\left(v_E-\omega h_0\right)^2R_E}. \qquad \textbf{0.1}$$

Numerical calculation gives$r_{\min}=0.43\,\text{AU}$, meaning the Mercury's orbit is within our reach. **0.1**

References

[1] Artsutanov, Y. Kosmos na elektrovoze. *Komsomolskaya Pravda* July 31 (1960); contents described in Lvov *Science* **158**, 946–947 (1967).

[2] Pearson, J. The Orbital Tower: a Spacecraft Launcher Using the Earth's Rotational Energy. *Acta Astronautica* **2**, 785 (1975)

[3] Aravind, P. K. The physics of the space elevator. *American Journal of Physics* **75**, 125 (2007).

[4] Bochníček, Z. A Carbon Nanotube Cable for a Space Elevator**.** *The Physics Teacher* **51**, 462 (2013).