Theory
Question 3: Tippe Top
Solutions

Reference sheet for markers

Note: some results below were used for the previous version of part **A.10**, and are no longer needed.

Coordinate systems for convenience (note: use of matrices not needed) xyz from XYZ

$$\begin{bmatrix}\hat{\mathbf{x}}\\ \hat{\mathbf{y}}\\ \hat{\mathbf{z}}\end{bmatrix} = \begin{bmatrix}\cos\phi & \sin\phi & 0\\ -\sin\phi & \cos\phi & 0\\ 0 & 0 & 1\end{bmatrix}\begin{bmatrix}\hat{\mathbf{X}}\\ \hat{\mathbf{Y}}\\ \hat{\mathbf{Z}}\end{bmatrix}$$

123 from xyz

$$\begin{bmatrix}\hat{\mathbf{1}}\\ \hat{\mathbf{2}}\\ \hat{\mathbf{3}}\end{bmatrix} = \begin{bmatrix}\cos\theta & 0 & -\sin\theta\\ 0 & 1 & 0\\ \sin\theta & 0 & \cos\theta\end{bmatrix}\begin{bmatrix}\hat{\mathbf{x}}\\ \hat{\mathbf{y}}\\ \hat{\mathbf{z}}\end{bmatrix}$$

Position of point A from centre of mass, in xyz and 123 frames:

$$\begin{aligned}\mathbf{a} &= \alpha R\hat{\mathbf{3}} - R\hat{\mathbf{z}} \qquad (1)\\ &= \alpha R\sin\theta\hat{\mathbf{x}} + R(\alpha\cos\theta - 1)\hat{\mathbf{z}}\\ &= R\sin\theta\hat{\mathbf{1}} + R(\alpha - \cos\theta)\hat{\mathbf{3}}\end{aligned}$$

Useful products:

$$\hat{\mathbf{z}}\times\hat{\mathbf{3}} = \sin\theta\hat{\mathbf{y}} \qquad (2)$$

$$(3)$$

Note (given in question):

$$\left(\frac{\partial\mathbf{A}}{\partial t}\right)_{\mathbf{K}} = \left(\frac{\partial\mathbf{A}}{\partial t}\right)_{\tilde{\mathbf{K}}} + \boldsymbol{\omega}\times\mathbf{A} \qquad (4)$$

Time derivatives:

$$\dot{\hat{\mathbf{3}}} = \boldsymbol{\omega}\times\hat{\mathbf{3}} \qquad (5)$$

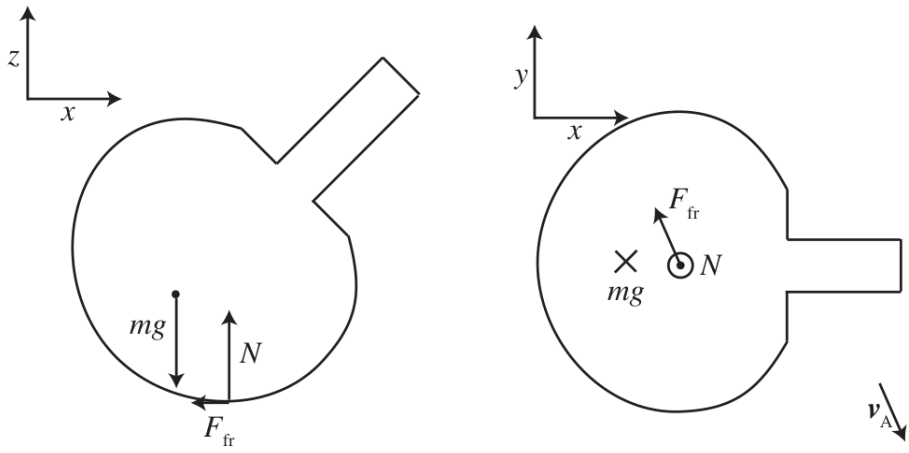
$$\dot{\hat{\mathbf{x}}} = \dot{\phi}\hat{\mathbf{y}} \qquad (6)$$

$$\dot{\hat{\mathbf{y}}} = -\dot{\phi}\hat{\mathbf{x}} \qquad (7)$$

Solutions: Tippe Top

1. (**1.0 marks**)

Free body diagrams:



Note: the direction of $\mathbf{F}_f$ must be opposite to the direction of $\mathbf{v}_A$, but is otherwise unimportant. Sum of forces:

$$\begin{aligned}\mathbf{F}_{\text{ext}} &= (N - mg)\hat{z} + \mathbf{F}_f \quad \text{(sufficient for full marks)} \\ &= (N - mg)\hat{z} - \frac{\mu_k N}{|v_A|}\mathbf{v_A}\end{aligned} \tag{8}$$

Sketched $\mathbf{v_A}$ must be in opposite direction to $\mathbf{F}_f$ on xy diagram.

2. (**0.8 marks**)

Sum of torques:

$$\begin{aligned}\boldsymbol{\tau}_{\text{ext}} &= \mathbf{a} \times (N\hat{\mathbf{z}} + \mathbf{F}_f) \qquad (9)\\ &= (\alpha R\hat{\mathbf{3}} - R\hat{\mathbf{z}}) \times (N\hat{\mathbf{z}} + F_{f,x}\hat{\mathbf{x}} + F_{f,y}\hat{\mathbf{y}}) \\ &= \alpha RN\hat{\mathbf{3}} \times \hat{\mathbf{z}} + \alpha R(\sin\theta\hat{\mathbf{x}} + \cos\theta\hat{\mathbf{z}}) \times (F_{f,x}\hat{\mathbf{x}} + F_{f,y}\hat{\mathbf{y}}) - R\hat{\mathbf{z}} \times (F_{f,x}\hat{\mathbf{x}} + F_{f,y}\hat{\mathbf{y}}) \\ &= -\alpha RN\sin\theta\hat{\mathbf{y}} + \alpha R\sin\theta F_{f,y}\hat{\mathbf{z}} + \alpha R\cos\theta F_{f,x}\hat{\mathbf{y}} - \alpha R\cos\theta F_{f,y}\hat{\mathbf{x}} - RF_{f,x}\hat{\mathbf{y}} + RF_{f,x}\hat{\mathbf{x}} \\ &= RF_{f,y}(1 - \alpha\cos\theta)\hat{\mathbf{x}} + [RF_{f,x}(\alpha\cos\theta - 1) - \alpha RN\sin\theta]\,\hat{\mathbf{y}} + \alpha R\sin\theta F_{f,y}\hat{\mathbf{z}} \qquad (10)\end{aligned}$$

3. (**0.4 marks**)

Motion at A satisfies

$$\mathbf{v_A} = \dot{\mathbf{s}} + \boldsymbol{\omega} \times \mathbf{a} \tag{11}$$

where $\boldsymbol{\omega}$ is the total angular velocity of the top in the centre of mass frame (this is deteremined in the next part). Want to show that $\mathbf{v_A} \cdot \hat{\mathbf{z}} = 0$.

To show this, take time derivative of contact condition in XYZ or xyz frame (note: either is suitable, as we only need the $\hat{\mathbf{z}}$ component, and $\hat{\mathbf{z}} = \hat{\mathbf{Z}}$).

Contact condition:

$$\begin{aligned}(\mathbf{s} + \mathbf{a}) \cdot \hat{\mathbf{z}} = 0 \quad &\text{at all times} \qquad (12)\\ \Rightarrow \frac{d}{dt}(\mathbf{s} + \mathbf{a}) \cdot \hat{\mathbf{z}} = 0 \quad &\text{at all times}\end{aligned}$$

Note we only care about the z-component, and $(\boldsymbol{\omega} \times \hat{\mathbf{z}}) \cdot \hat{\mathbf{z}} = 0$. Then, using 11, 1, and 5,

$$\begin{aligned}\mathbf{v_A} \cdot \hat{\mathbf{z}} &= (\dot{\mathbf{s}} + \boldsymbol{\omega} \times \mathbf{a}) \cdot \hat{\mathbf{z}} \\ &= \left(\dot{\mathbf{s}} + \alpha R\boldsymbol{\omega} \times \hat{\mathbf{3}}\right) \cdot \hat{\mathbf{z}} \\ &= \left(\dot{\mathbf{s}} + \alpha R\frac{d\hat{\mathbf{3}}}{dt}\right) \cdot \hat{\mathbf{z}} \\ &= (\dot{\mathbf{s}} + \dot{\mathbf{a}}) \cdot \hat{\mathbf{z}} = 0 \qquad (13)\end{aligned}$$

4. (**0.8 marks**)

Total angular velocity $\boldsymbol{\omega}$ of top is the sum of three distinct rotations:

$$\boldsymbol{\omega} = \dot{\theta}\hat{\mathbf{2}} + \dot{\phi}\hat{\mathbf{z}} + \dot{\psi}\hat{\mathbf{3}}$$

Use transformations shown in figure 3 or otherwise to transform into xyz or 123 frame:

$$\boldsymbol{\omega} = \dot{\psi}\sin\theta\hat{\mathbf{x}} + \dot{\theta}\hat{\mathbf{y}} + (\dot{\psi}\cos\theta + \dot{\phi})\hat{\mathbf{z}} \tag{14}$$

$$\boldsymbol{\omega} = -\dot{\phi}\sin\theta\hat{\mathbf{1}} + \dot{\theta}\hat{\mathbf{2}} + (\dot{\psi} + \dot{\phi}\cos\theta)\hat{\mathbf{3}} \tag{15}$$

5. (**1.0 marks**)

Where $\mathbf{I}$ is the inertia tensor

$$\begin{bmatrix} I_1 & 0 & 0 \\ 0 & I_1 & 0 \\ 0 & 0 & I_3, \end{bmatrix}$$

we have

$$\begin{aligned} E_T &= K_T + K_R + U_G \\ &= \frac{1}{2}\boldsymbol{\omega}\cdot\mathbf{I}\boldsymbol{\omega} + \frac{1}{2}m\dot{\mathbf{s}}^2 + mgR(1-\alpha\cos\theta) \end{aligned}$$

From 11,

$$\begin{aligned} \dot{\mathbf{s}} &= \mathbf{v_A} - \boldsymbol{\omega}\times\mathbf{a} \\ &= \mathbf{v_A} - (\dot\theta\hat{\mathbf{2}} + \dot\phi\hat{\mathbf{z}} + \dot\psi\hat{\mathbf{3}})\times(\alpha R\hat{\mathbf{3}} - R\hat{\mathbf{z}}) \\ &= v_x\hat{\mathbf{x}} + v_y\hat{\mathbf{y}} - \left(\dot\theta\alpha R\hat{\mathbf{1}} - \dot\theta R\hat{\mathbf{z}} + \dot\phi\alpha R\hat{\mathbf{z}}\times\hat{\mathbf{3}} - \dot\psi R\hat{\mathbf{3}}\times\hat{\mathbf{z}}\right) \\ &= \left(v_x + \dot\theta R(1-\alpha\cos\theta)\right)\hat{\mathbf{x}} + \left(v_y - R\sin\theta(\alpha\dot\phi+\dot\psi)\right)\hat{\mathbf{y}} + \dot\theta\alpha R\sin\theta\hat{\mathbf{z}} \end{aligned}$$

using 2. Thus

$$\begin{aligned} E_T = \frac{1}{2}\left[I_1(\dot\phi^2\sin^2\theta + \dot\theta^2) + I_3(\dot\psi + \dot\phi\cos\theta)^2\right] \\ + \frac{m}{2}\left[\left(v_x + \dot\theta R(1-\alpha\cos\theta)\right)^2 + \left(v_y - R\sin\theta(\alpha\dot\phi+\dot\psi)\right)^2 + \dot\theta^2\alpha^2R^2\sin^2\theta\right] + mgR(1 \end{aligned}$$

6. (**0.4 marks**)

From 10,

$$\frac{d\mathbf{L}}{dt}\cdot\hat{\mathbf{z}} = \sum\boldsymbol{\tau}\cdot\hat{\mathbf{z}} = \alpha R\sin\theta F_{f,y} \tag{16}$$

7. (**1.4 marks**)

Changes in energy: $h = \mathbf{s}\cdot\hat{\mathbf{z}}$ increases, so $\dot U_G > 0$.

At start and end (phases **I** and **V**) there is little translation so $K_T \sim 0$ at **I** and **V**. Thus, energy transfer is from K_R to U_G.

Normal force does no work. Frictional force does work at point A. Direction is $-\mathbf{v_A}$:

$$\begin{aligned} W &= \int \mathbf{F}_f\cdot\mathbf{v_A}\ dt < 0 \\ \Rightarrow \frac{d}{dt}E_T &= -\mu_k N|\mathbf{v_A}| \end{aligned}$$

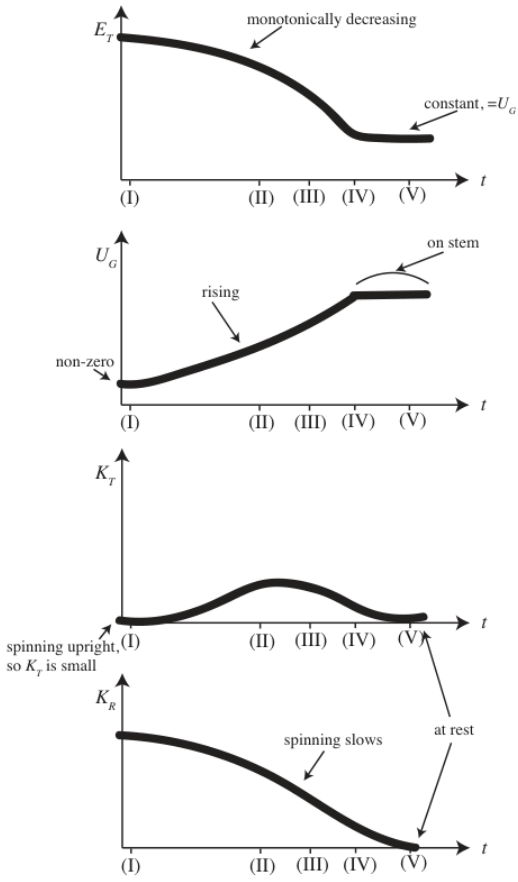
Thus $\mathbf{F}_f$ decreases the total energy monotonically.

16 implies only the $\mathbf{F}_f.\hat{\mathbf{y}}$ acts to decrease $\mathbf{L}\cdot\hat{\mathbf{z}}$. Energy transfer from K_R to U_G, caused by component of frictional force in $\hat{\mathbf{y}}$ direction, so component of resultant torque is in the $\mathbf{a}\times\hat{\mathbf{y}}$ direction.

8. (**2.0 marks**)

Expectation (see figure):

- E_T: monotonically decreasing
- K_R: monotonically decreasing; zero at V
- K_T: zero at I and V; higher between; close to zero at IV
- U_G: flat at start and finish; higher at end; increases from I to IV then flat; increase roughly at same time that K_{rot} decreases



9. (**0.5 marks**)

From 15,

$$\mathbf{L} = \mathbf{I}\boldsymbol{\omega} = I_1\left(-\dot{\phi}\sin\theta\hat{\mathbf{1}} + \dot{\theta}\hat{\mathbf{2}}\right) + I_3(\dot{\psi} + \dot{\phi}\cos\theta)\hat{\mathbf{3}} \tag{17}$$

Taking cross product with $\hat{\mathbf{3}}$:

$$\begin{aligned}\mathbf{L}\times\hat{\mathbf{3}} &= I_1\left(\dot{\phi}\sin\theta\hat{\mathbf{2}} + \dot{\theta}\hat{\mathbf{1}}\right)\\ &= I_1(\boldsymbol{\omega}\times\hat{\mathbf{3}})\end{aligned} \tag{18}$$

10. (**1.7 marks**)

About any axis through the centre of mass,

$$\frac{d\mathbf{L}}{dt} \neq 0 \Leftrightarrow \tau_{\text{ext}} \neq 0$$

External torque given by 9,

$$\begin{aligned}\boldsymbol{\tau}_{\text{ext}} &= \mathbf{a}\times(N\hat{\mathbf{z}} + \mathbf{F}_f)\\ \Rightarrow \tau_{\text{ext}}\cdot\mathbf{a} &= 0\\ \frac{d\mathbf{L}}{dt}\cdot\mathbf{a} &= 0\end{aligned}$$

Thus, angular momentum in the direction of $\mathbf{a}$ must be constant, so $\mathbf{v} = \mathbf{a}$.

To demonstrate this mathematically, 5, 10, 18 allow

$$\begin{aligned}-\dot{\lambda} &= \frac{d\mathbf{L}}{dt}\cdot\mathbf{a} + \alpha R\mathbf{L}\cdot\frac{d\hat{\mathbf{3}}}{dt}\\ &= (\mathbf{a}\times(N\hat{\mathbf{z}} + \mathbf{F_f}))\cdot\mathbf{a} + \frac{\alpha R}{I_1}\mathbf{L}\cdot(\boldsymbol{\omega}\times\mathbf{L})\\ &= 0\end{aligned}$$