Atmospheric Physics (10.0 points)

The Earth's atmosphere is a complex physical system, and predicting its behavior is crucial for environmental and meteorological purposes. However, even the best theoretical models run on modern computers are insufficient to make precise predictions. In this problem, we will attempt to understand some of the basic atmospheric phenomena based on simple models. You might need the following constants: the mean solar power per unit area at Earth, the total solar irradiance $F_s = 1370\,\mathrm{W/m^2}$, molar mass of water $\mu_{H_2O} \approx 18\,\mathrm{g/mol}$ and average molar mass of air $\mu_{\mathrm{air}} \approx 29\,\mathrm{g/mol}$, The Stefan-Boltzmann constant $\sigma = 5.67 \times 10^{-8}\,\mathrm{W/(m^2K^4)}$. All gases in this problem can be treated as ideal gases. Assume that all air molecules have 5 degrees of freedom. You may need the following integral:

$$\int_{-\infty}^{\infty} e^{-ax^2/2}\,dx = \sqrt{\frac{2\pi}{a}}, \quad a > 0.$$

Part A. Surface Temperature of the Earth (1.2 points)

In this section, we study the effect of the atmosphere on the Earth surface's temperature. Assume that Earth and its atmosphere have an albedo $a = 0.3$ for solar radiation, which is the reflected fraction of the total incident radiation. You may use this value in all parts of this problem. In addition, assume the Earth radiates as a black body.

A.1 Express the average net solar power received by the Earth and atmosphere system P_0 in terms of F_s, a and R_E, the radius of the Earth. 0.2 pt

A.2 Estimate the temperature of the Earth's surface T_{g0} assuming that it is at a steady state. Ignore the atmosphere. 0.3 pt

Your answer for **A.2** should be lower than what you would expect. We now consider adding a thin atmospheric layer at temperature T_a, see *Figure A.1*. The atmospheric layer transmits a net fraction t_{sw} of the incident solar radiation and a net fraction t_{lw} of the Earth's thermal radiation. Otherwise, you may treat the atmosphere as a black body.

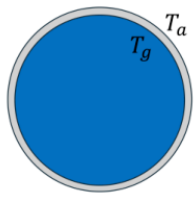


Figure A.1

A.3 Assuming the system is in a steady state, calculate T_g, the temperature of the ground. Use $t_{\mathrm{sw}} = 0.9$ and $t_{\mathrm{lw}} = 0.2$. 0.7 pt

Part B. The absorption spectrum of atmospheric gases (1.8 points)

The infrared radiation emitted by Earth has low energy, incapable of exciting electrons within the molecules, but it has the ability to excite the vibrational and rotational modes of the molecules.

B.1 Consider a simple diatomic molecule modeled as two point masses m_A and m_B connected by a spring with spring constant k. What is the angular frequency of vibrations ω_d? 0.5 pt

B.2 Quantum mechanics dictates that vibrational excitations due to absorbing a photon can only raise the quantum energy level by one. What is the energy of the photon E_p that can excite the vibration in **B.1**? Neglect recoil effects. 0.2 pt

Quantum mechanics forbids the vibrational modes of symmetric diatomic molecules, such as nitrogen and oxygen (the most abundant gasses in the Earth's atmosphere) to be excited by light. This explains why N_2 and O_2 do not contribute to the green

house effect. In general, the absorption of light by molecules is governed by the allowed energy transitions in them. However, the energy of the light absorbed does not have to exactly match the energy gap in the molecule. Suppose that a molecule at rest has a spectral line (an allowed transition) at frequency f_0.

B.3 What is the shift in the spectral line $f - f_0$ if the molecule is moving with velocity v towards the emitter such that $|v| \ll c$, where c is the speed of light. 0.2 pt

For a gas at temperature T, the velocity of its molecules is distributed according to Maxwell's distribution. For a molecule of mass m, the probability to find a molecule's velocity along one dimension to be between v and $v + dv$ is $p_1(v)dv$, where $p_1(v)$ is a probability distribution function given by

$$p_1(v) = C \exp\left(-\frac{mv^2}{2k_B T}\right)$$

C is a normalization constant ensuring the probabilities add up to one, and k_B is the Boltzmann constant.

B.4 Find the normalization constant C, assuming that the velocity v could range from $-\infty$ to ∞. 0.2 pt

B.5 Find the probability distribution function $p_2(f)$ to find a molecule with a spectral line f_0 shifted to f due to thermal motion, up to a normalization factor, in terms of f, f_0, T, m and fundamental constants. 0.3 pt

B.6 Sketch $p_2(f)$ as a function of $f - f_0$, and determine the shift $f^\star - f_0$ at which $p_2(f^\star)$ is a fraction $1/e$ of its peak value, where e is the natural number. 0.4 pt

Part C. Stability of air in the atmosphere (2.7 points)

Consider a small cylindrical mass of air at height z above the ground. The pressure and mass density of air at that height are $p(z)$ and $\rho(z)$, respectively, see *Figure C.1.* Assume a uniform downward gravitational field g and that the pressure on the Earth's surface is p_o.

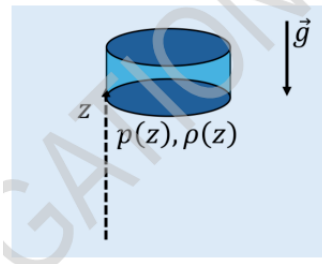


Figure C.1

C.1 Assuming that the small air mass is at hydrostatic equilibrium, derive an expression of the rate of change of pressure with respect to height, dp/dz in terms of g and $\rho(z)$. 0.3 pt

C.2 Express dp/dz in terms of $\mu_{\text{air}}, g, p(z)$ and $T(z)$, the temperature at height z and fundamental constants. 0.2 pt

C.3 Assuming an isothermal atmosphere, $T(z) = T$, find an expression for $p(z)$ in terms of $z, \mu_{\text{air}}, g, p_o, T$ and fundamental constants. 0.2 pt

In a real atmosphere, the temperature is not constant but changes with height. The rate of decrease of temperature with height $\Gamma(z) = -dT/dz$ is called the lapse rate. Consider a small mass of air rising adiabatically in the atmosphere such that it remains at mechanical equilibrium with its surrounding.

C.4 For the adiabatically rising air mass, find the adiabatic lapse rate Γ_a in terms of c_p, the molar specific heat at constant pressure, μ_{air} and g. 0.6 pt

To analyze the stability of an atmosphere, we imagine starting from an equilibrium state, and then perturbing a small mass of air and analyze its response. Consider

a small air mass initially in equilibrium with the surrounding air at height z and temperature T. It is then adiabatically displaced vertically by a displacement δz_0. Assume that throughout the motion, the air parcel always has the same pressure as the surrounding air at the same height. The surrounding atmosphere is unaltered and has a different lapse rate Γ. Neglect viscosity.

C.5 Find the equation of motion for δz, the instantaneous vertical displacement. Under what condition is the equilibrium at z stable? What is the angular frequency ω of small oscillation? Express your answers in terms of $T, \Gamma, g, \mu_{\text{air}}$ and c_p. 1.4 pt

Part D. Moisture (2.7 points)

Even though water constitutes a small portion of the atmosphere, it has a significant role in climate science. It is responsible for precipitation, and it is the most significant greenhouse gas. The phase of water depends on what temperature and pressure the water system is at, depicted on a $p - T$ phase diagram, see *Figure D.1.* When the pressure and temperature lie on the coexistence curve, both liquid and vapor water can be present in the system. The slope of the coexistence curve is given by the Clausius-Clapeyron equation:

$$\frac{dp_s}{dT} = \frac{\Delta S}{\Delta V}$$

where p_s is the saturation pressure, the pressure at the phase transition, ΔS and ΔV are the changes in entropy and volume across the phase transitions, respectively. Treat water vapor as an ideal gas.

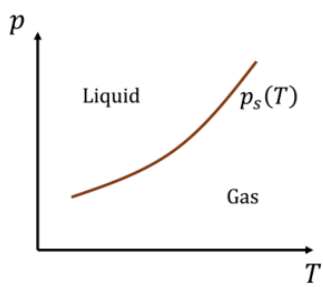


Figure D.1

D.1 Express dp_s/dT for the water liquid-vapor coexistence curve in terms of the water latent heat of evaporation $L, \mu_{\text{H}_2\text{O}}, p_s, T$ and fundamental constants. 0.5 pt

D.2 If for some reference temperature T_o, $p_s = p_{so}$, find an expression for $p_s(T)$ in terms of $p_{so}, \mu_{\text{H}_2\text{O}}, L, T, T_o$ and fundamental constants. 0.2 pt

Now we consider a `moist' air mass that rises adiabatically starting from a temperature T_i. The mass mixing ratio of water vapor (the mass of water vapor relative to the total mass) is ϕ. Take the air mass to have a specific molar heat at constant pressure c_p. The universal gas constant is $R = 8.31\,\text{J/(mol K)}$.

D.3 Assuming that the air mass starts at $T_i = 17.0°\text{C}$ and $p_i = 10^5\,\text{Pa}$. Find the temperature T_l at which liquid water starts forming in it if $\phi = 10^{-2}$. Assume that the water content in the air mass stays constant during the rise. Use $L = 2460\,\text{kJ/kg}$ and $p_{so} = 1.94 \times 10^3\,\text{Pa}$ at $T_i = 17.0°\text{C}$. 2.0 pt

Part E. Sun halo (1.6 points)

Under suitable atmospheric conditions, a bright ring appears around the Sun which is called a halo. Halos are caused by ice crystals present in the upper troposphere. One interesting feature about halos is that they always appear at a specific angle relative to the direction of the Sun.



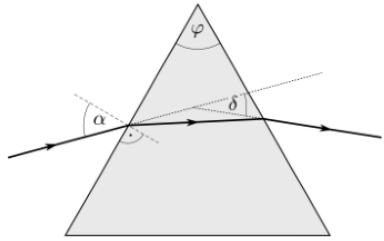


Figure E.1. On the left: A photograph showing a halo around the Sun. On the right: The path of a light ray passing through the prism.

E.1 Consider a simple prism with an apex angle of φ and direct a light ray onto it at an incidence angle α, as shown in *Figure E.1.* Let the refractive index of the prism be n. Express the angle of deviation δ of the light ray after passing through the prism in terms of α, n and φ. 0.8 pt

The most common type of halo forms when tiny ice crystals take the shape of regular hexagonal prisms. Light from the Sun falls onto randomly oriented ice crystals drifting in the atmosphere and scatters into various directions. However, in certain specific directions, the intensity of the refracted light is maximal, and this determines the angle at which the bright ring appears.

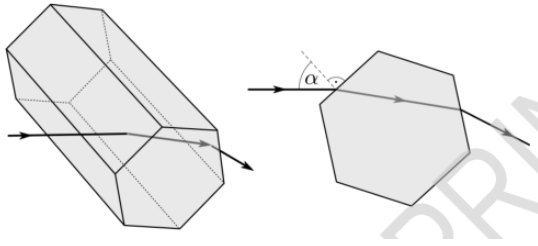


Figure E.2.

Consider a hexagonal ice prism whose six-fold symmetry axis is perpendicular to the direction of the Sun's rays. Investigate a light ray that refracts through two rectangular faces of the prism indicated in *Figure E.2.* Due to the random orientation of the ice crystals, the light strikes the crystal faces at varying incidence angles α.

E.2 Plot on the answer sheet how the deviation angle δ of the examined light ray depends on the incidence angle α within the interval $[20°, 70°]$ in $5°$ increments. The refractive index of ice is n=1.31. 0.6 pt

E.3 Using the graph from the previous part, determine at what angle the halo appears the brightest relative to the direction of the Sun. 0.2 pt