T2. Waves and Phase Transitions in Spin Systems (10.0 pts)

Part A. Precession and interactions of magnetic dipoles (1.2 points)

A.1: The angular momentum of the planar loop rotating around its perpendicular axis is given by

$$\vec{L} = MR^2\vec{\omega},$$

while the current generated by the rotation is $I = Q/T = \omega Q/2\pi$, then the magnetic dipole moment of the planar loop is given by

$$\vec{\mu} = IA\hat{\omega} = \frac{\omega Q}{2\pi}\pi R^2\hat{\omega} = \frac{Q}{2}R^2\vec{\omega}.$$

It follows that

$$\vec{\mu} = \frac{Q}{2M}\vec{L}$$

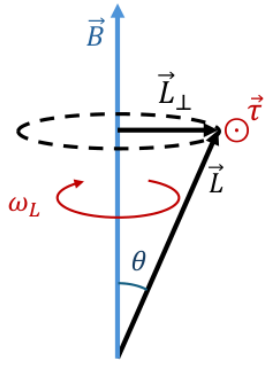
and

$$\gamma = \frac{Q}{2M}.$$

Grading scheme for Task A.1.	**Pts**
correct result for L (just magnitude is fine)	0.1
correct result for μ (just magnitude is fine)	0.1
correct result for γ	0.1
Total	**0.3**

A.2: The torque acting on a magnetic dipole due to an external magnetic field is given by

$$\vec{\tau} = \vec{\mu} \times \vec{B} = \frac{\mathrm{d}\vec{L}}{\mathrm{d}t}.$$



Since only the component perpendicular to $\vec{B}$ can change, and it rotates with angular velocity ω_L, we write

$$\Rightarrow L\omega_L \sin\theta = -\mu B \sin\theta$$

from which we deduce

$$\omega_L = -\frac{\mu}{L}B = -\gamma B.$$

Note: The torque equation can be found by drawing an analogy to the electric dipole case.

Grading scheme for Task A.2.	**Pts**
Writing the torque formula	0.1
Realizing only the perpendicular component $L\sin\theta$ is changing	0.1
Correct magnitude for ω_L	0.1
Correct sign for ω_L	0.1
Total	**0.4**

Grading note: If the final answer for ω_L is written without justification, a maximum of 0.1 is given.

A.3: The magnetic field due to the first dipole on the second is

$$\vec{B}_{1 \text{ on } 2} = -\frac{\mu_0}{4\pi}\frac{\vec{\mu}_1}{d^3}.$$

Then

$$U = -\vec{\mu}_2 \cdot \vec{B}_{1 \text{ on } 2} = -\frac{\mu_0}{4\pi d^3}\mu_1\mu_2\cos(\pi - \theta)$$

leading to

$$U = \frac{\mu_0\gamma^2}{4\pi d^3}\vec{L}_1 \cdot \vec{L}_2$$

and

$$J_0 \equiv \frac{\mu_0\gamma^2}{4\pi d^3}.$$

Grading scheme for Task A.3.	**Pts**
Writing the interaction energy correctly	0.1
Writing the correct magnetic field magnitude	0.1
Writing the correct magnetic field direction	0.1
Correct magnitude for J_0	0.1
Correct sign for J_0	0.1
Total	**0.5**

Grading note: If the direction of magnetic field is not explicitly drawn or written, then credit is given if and only if the sign for interaction energy AND J_0 are correct.

Part B. Spin waves (4.5 points)

B.1: The ith spin interacts with the $i-1$ and $i+1$ spins, with an energy $E_i = -J(\vec{S}_{i-1} + \vec{S}_{i+1}) \cdot \vec{S}_i$ which is analogous to $E_i = -\vec{B}_i \cdot \vec{\mu}_i$. Using $\vec{\mu}_i = \gamma\vec{S}_i$, we find

$$\vec{B}_{i,\text{eff}} = \frac{J}{\gamma}\left(\vec{S}_{i-1} + \vec{S}_{i+1}\right).$$

Grading scheme for Task B.1.	**Pts**
Explicit understanding that $i-1$ and $i+1$ are the contributors for spin i	0.2
Correct result for effective magnetic field	0.1
Total	**0.3**

Grading note: Writing the correct result directly receives full credit.

B.2: Given the effective magnetic field

$$\begin{aligned}\frac{\mathrm{d}\vec{S}_i}{\mathrm{d}t} &= \vec{\tau} = \vec{\mu}_i \times \vec{B}_{i,\text{eff}} \\ &= J\vec{S}_i \times \left(\vec{S}_{i-1} + \vec{S}_{i+1}\right).\end{aligned}$$

Grading scheme for Task B.2.	**Pts**
Writing the rate of change of $\vec{S}$ using the effective magnetic field	0.1
Correct equation	0.2
Total	**0.3**

Grading note: No partial credit on this Task.

B.3: We can write the rate of change of the x and y components of $\vec{S}_i$, keeping in mind the approximation $S_{i,z} \approx S$ for all i.

$$\begin{aligned}\frac{\mathrm{d}S_{i,x}}{\mathrm{d}t} &= J\left[S_{i,y}\left(S_{i-1,z}+S_{i+1,z}\right)\right. \\ &\quad \left.-S_{i,z}\left(S_{i-1,y}+S_{i+1,y}\right)\right] \\ &\approx JS\left(2S_{i,y}-S_{i-1,y}-S_{i+1,y}\right)\end{aligned}$$

and

$$\begin{aligned}\frac{\mathrm{d}S_{i,y}}{\mathrm{d}t} &= J\left[-S_{i,x}\left(S_{i-1,z}+S_{i+1,z}\right)\right. \\ &\quad \left.-S_{i,z}\left(S_{i-1,x}+S_{i+1,x}\right)\right] \\ &\approx -JS\left(2S_{i,x}-S_{i-1,x}-S_{i+1,x}\right).\end{aligned}$$

The structure of these equations along with the traveling wave behavior leads us to the ansatz

$$\begin{aligned}S_{i,x} &= \delta S\cos(kx-\omega t) \\ S_{i,y} &= \delta S\sin(kx-\omega t),\end{aligned}$$

Where δS is the amplitude. This yields

$$\begin{aligned}\omega\delta S\sin(kx-\omega t) = JS\delta S\cdot[&2\sin(kx-\omega t) \\ &-\sin(kx-\omega t-ka) \\ &-\sin(kx-\omega t+ka)].\end{aligned}$$

Using $\sin(A)+\sin(B) = 2\sin\left(\frac{A+B}{2}\right)\cos\left(\frac{A-B}{2}\right)$, we get

$$\omega(k) = 2JS\left[1-\cos(ka)\right].$$

Note: We can show that the amplitude for S_x equals that of S_y, but substituting predictions with δS_x and δS_y, which leads to

$$\begin{aligned}\omega\delta S_x\sin(kx-\omega t) = JS\delta S_y\cdot[&2\sin(kx-\omega t) \\ &-\sin(kx-\omega t-ka) \\ &-\sin(kx-\omega t+ka)] \\ \omega\delta S_y\cos(kx-\omega t) = JS\delta S_x\cdot[&2\cos(kx-\omega t) \\ &-\cos(kx-\omega t-ka) \\ &-\cos(kx-\omega t+ka)]\end{aligned}$$

which can only be satisfied given $\delta S_x = \delta S_y$.

Grading scheme for Task B.3.	**Pts**
Writing traveling waves as function of $kx \pm \omega t$ (either sign is okay, either trig function or complex exponentials is okay)	0.25
Same amplitude for S_x and S_y	0.25
Correct phase relation between S_x and S_y (a difference of $\pi/2$)	0.25
Writing the correct explicit equation of motion for either S_x or S_y	0.5
Explicit use of $S_z \approx S$	0.25
Correct final result ($\pm$ is okay)	0.5
Total	**2.0**

Grading note 1: credit should be given to the amplitudes equality even if not proved. *Grading note 2*: In case the student does not arrive at the correct final solution, but has made a serious attempt at the algebra involving trig identities or complex exponentials, then up to 0.2 points may be rewarded.

B.4: For small k,

$$\omega(k) \approx 2JS\left[1 - 1 + \frac{1}{2}(ka)^2\right] = JSa^2k^2$$

The de Broglie relations are $E = \hbar\omega$ and $p = \hbar k$, plugging these into the expression for $\omega(k)$, we find

$$E = \hbar\omega = \frac{JSa^2}{\hbar}p^2 \equiv \frac{p^2}{2m_{\text{eff}}},$$

where

$$m_{\text{eff}} \equiv \frac{\hbar}{2JSa^2}.$$

Grading scheme for Task B.4.	**Pts**
Correct Taylor expansion result	0.2
Correct relation between momentum and wave vector	0.1
Correct relation between energy and angular frequency	0.1
Correct identification of effective mass	0.2
Total	**0.6**

Grading notes: If the student did not find the correct $\omega(k)$, but states that a massive particle has energy $E = p^2/2m$, then give 0.1 points (with a max of 0.3 given both de Broglie relations).

B.5: In this inelastic scattering, energy and momentum for the entire system (including the chain) is conserved. In particular, the spin wave does not have a momentum along the y-axis. Therefore

$$p_{\text{in}}\cos\theta_{\text{in}} = p_{\text{out}}\cos\theta_{\text{out}}.$$

Combining this with $E_n = p_n^2/2m_n$ valid for the neutron, we find

$$E_{\text{out}} = \left(\frac{\cos\theta_{\text{in}}}{\cos\theta_{\text{out}}}\right)^2 E_{\text{in}}$$

Using energy conservation, the energy E_s of the spin wave is $E_s = E_{\text{in}} - E_{\text{out}}$, so we get

$$E_s = \frac{\cos^2\theta_{\text{out}} - \cos^2\theta_{\text{in}}}{\cos^2\theta_{\text{out}}}E_{\text{in}}.$$

Conservation of momentum along the x-axis gives

$$\begin{aligned} p_s &= p_{\text{in}}\sin\theta_{\text{in}} - p_{\text{out}}\sin\theta_{\text{out}} \\ &= \sqrt{2m_nE_{\text{in}}}\left(\sin\theta_{\text{in}} - \frac{\cos\theta_{\text{in}}}{\cos\theta_{\text{out}}}\sin\theta_{\text{out}}\right). \end{aligned}$$

Then from $m_{\text{eff}} = p_s^2/2E_s$, we find

$$m_{\text{eff}} = \frac{\cos^2\theta_{\text{out}}\left(\sin\theta_{\text{in}} - \frac{\cos\theta_{\text{in}}}{\cos\theta_{\text{out}}}\sin\theta_{\text{out}}\right)^2}{\cos^2\theta_{\text{out}} - \cos^2\theta_{\text{in}}}m_n$$

which after simplifying gives

$$m_{\text{eff}} = \frac{\sin^2(\theta_{\text{in}} - \theta_{\text{out}})}{\cos^2\theta_{\text{out}} - \cos^2\theta_{\text{in}}}m_n = \frac{\sin(\theta_{\text{in}} - \theta_{\text{out}})}{\sin(\theta_{\text{in}} + \theta_{\text{out}})}m_n.$$

Grading scheme for Task B.5.	Pts
Conservation of momentum along y-axis	0.4
Conservation of momentum along x-axis	0.3
Conservation of energy	0.2
Relation between E_{out} and E_{in}	0.1
Correct final answer (should be in either two forms at the end, otherwise 0.2)	0.3
Total	**1.3**

Part C. Phase transitions in spin chains (4.3 points)

C.1: In a Boltzmann distribution, given the system's temperature T, the probability to find a system in a given state with energy ε_i is

$$p_i \propto \exp\left(-\frac{\varepsilon_i}{k_B T}\right).$$

In this case, the probability to find a spin up state of energy $\varepsilon_\uparrow = -hs_\uparrow = -h$ is

$$p_\uparrow \sim \mathrm{e}^{h/k_B T}.$$

Therefore,

$$\frac{p_\uparrow}{p_\downarrow} = \frac{\mathrm{e}^{h/k_B T}}{\mathrm{e}^{-h/k_B T}} = \mathrm{e}^{2h/k_B T}.$$

We also note that, normalization requires, $p_\uparrow + p_\downarrow = 1$. Thus, a full solution for $p_\uparrow$ and $p_\downarrow$ is

$$p_\uparrow = \frac{\mathrm{e}^{h/k_B T}}{\mathrm{e}^{h/k_B T} + \mathrm{e}^{-h/k_B T}}, \tag{1}$$

$$p_\downarrow = \frac{\mathrm{e}^{-h/k_B T}}{\mathrm{e}^{h/k_B T} + \mathrm{e}^{-h/k_B T}}. \tag{2}$$

Grading scheme for Task C.1.	Pts
Use of Boltzmann factor	0.2
Correct Boltzmann factor for up and down (0.1 each)	0.2
Correct final result for the ratio	0.1
Total	**0.5**

Grading note 1: if the ratio is written immediately without writing the Boltzmann factors separately then give the 0.2 associated with that.

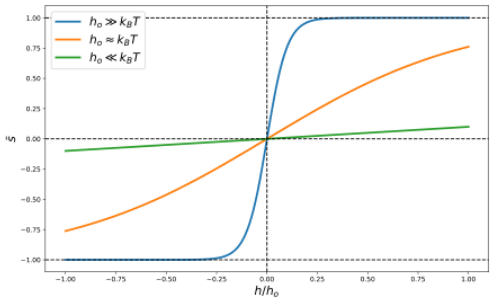
Grading note 2: even if the final answer is written immediately, the student must write $p \propto \exp(-E/k_B T)$ to receive full credit.

C.2: The average polarization of the system $\bar{s}$ can be written as

$$\begin{aligned}\bar{s} &= \frac{1}{N}\sum_i s_i, \\ &= \frac{1}{N}[Np_\uparrow \cdot 1 + Np_\downarrow \cdot (-1)], \\ &= p_\uparrow - p_\downarrow,\end{aligned}$$

where $Np_\uparrow$ and $Np_\downarrow$ are the number of spin vectors pointing up and down, respectively. Substituting for probabilities, we find

$$\bar{s} = \frac{\mathrm{e}^{h/k_B T} - \mathrm{e}^{-h/k_B T}}{\mathrm{e}^{h/k_B T} + \mathrm{e}^{-h/k_B T}} = \tanh\left(\frac{h}{k_B T}\right). \tag{3}$$



Grading scheme for Task C.2.	**Pts**
Deducing $\bar{s} = p_\uparrow - p_\downarrow$	0.2
Normalization condition $p_\uparrow + p_\downarrow = 1$	0.1
Final result for $\bar{s}$	0.1
Correct sketches (0.2 each)	0.6
Total	**1.0**

Grading note: For a sketch to be correct, it has to include labels of both axes and has intercept at $(0, 0)$. For $h_o \gg k_B T$, $\bar{s}$ has to approach ± 1 quickly. For $h_o \ll k_B T$, it should look like a line with tiny slope. For $h_o \approx k_B T$, it should be in-between the two cases.

C.3: The energy of the system is minimized when all the spins are aligned, so

$$E_g = -\tilde{J} \sum_i 1 = -\tilde{J}(N-1) \approx -\tilde{J}N, \tag{4}$$

where we used $N \gg 1$.

Grading scheme for Task C.3.	**Pts**
Realizing the spins minimize their energy by aligning along the same direction	0.1
Correct result (both $N-1$ and N are fine)	0.1
Total	**0.2**

C.4:

$$E = -\tilde{J} \sum_i s_i s_{i+1} = -\tilde{J} \sum_i s_i \bar{s} = -\tilde{J}_{\text{eff}} \sum_i s_i,$$

where we define $\tilde{J}_{\text{eff}} = \tilde{J}\bar{s}$. Using $2\bar{s}$ is double counting the energy.

Grading scheme for Task C.4.	**Pts**
Realizing s_{i+1} can be replaced with $\bar{s}$	0.1
Correct final result	0.1
Total	**0.2**

Grading note: If the student uses $2\bar{s}$, then no partial credit. However, no propagating error from this specific mistake for the following parts.

C.5: By looking at the earlier equation given in part C.4, and comparing it to what we had earlier in C.2, we find that the average polarization $\bar{s}$ should satisfy the following transcendental equation:

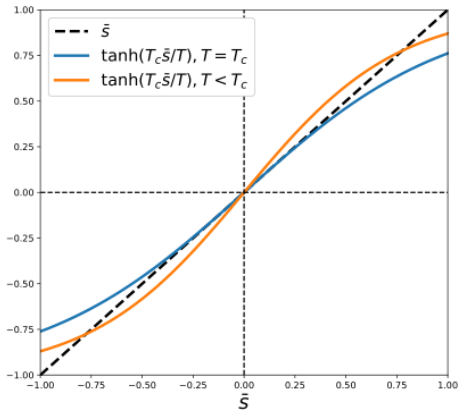
$$\bar{s} = \tanh\left(\frac{\tilde{J}_{\text{eff}}}{k_B T}\right) = \tanh\left(\frac{\tilde{J}\bar{s}}{k_B T}\right).$$

With the help of the plots we did earlier in C.2, one can see that for $\tilde{J} \ll k_B T$, there exists only one simple solution, $\bar{s} = 0$. Where as for $\tilde{J} \gg k_B T$, there exist two non-trivial solutions. As clarified by the figure below, this crossing behavior occurs when

the tangent line for $\tanh\left(\tilde{J}\bar{s}/k_BT\right)$ at $\bar{s}=0$ equals the slope of $\bar{s}$:

$$\begin{aligned}\frac{d}{d\bar{s}}\tanh\left(\frac{\tilde{J}\bar{s}}{k_BT_c}\right)\Big|_{\bar{s}=0} &= \frac{d}{d\bar{s}}\bar{s}\Big|_{\bar{s}=0}\\ \Rightarrow \frac{\tilde{J}}{k_BT_c}\frac{1}{\cosh^2\left(\tilde{J}\bar{s}/k_BT_c\right)}\Big|_{\bar{s}=0} &= 1\\ \Rightarrow \frac{\tilde{J}}{k_BT_c} &= 1,\end{aligned}$$

leading to $T_c = \tilde{J}/k_B$.



Grading scheme for Task C.5.	**Pts**
stating $\bar{s} = \tanh\left(\tilde{J}_{\text{eff}}/k_BT\right)$	0.1
Replacing $\tilde{J}_{\text{eff}}$ for h in $\bar{s}$ results from C.2	0.2
Realizing that there is one trivial solution for $\tilde{J} \ll k_BT$	0.2
Realizing that there are two non-trivial solutions for $\tilde{J} \gg k_BT$	0.2
Clearly stating the condition at when number of solutions changes	0.3
Correct final result for T_c	0.2
Total	**1.2**

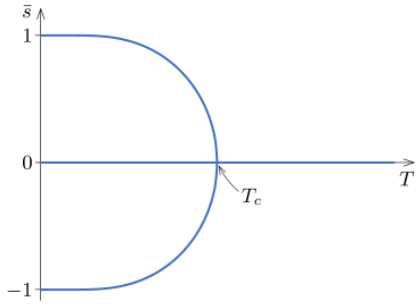
C.6: Near the critical temperature, the average polarization is small. Thus, we can approximate the transcendental equation as

$$\bar{s} = \tanh\left(\frac{T_c}{T}\bar{s}\right) \approx \frac{T_c}{T}\bar{s} - \frac{1}{3}\left(\frac{T_c}{T}\bar{s}\right)^3.$$

We see that $\bar{s}=0$ is still a solution. After rearranging for $\bar{s}\neq 0$, we get

$$\begin{aligned}\bar{s} &= \pm\sqrt{3\left[\left(\frac{T}{T_c}\right)^2 - \left(\frac{T}{T_c}\right)^3\right]}\\ &= \pm\sqrt{3\left(\frac{T}{T_c}\right)^2\left(1-\frac{T}{T_c}\right)}\\ &\approx \pm\sqrt{3\frac{T_c-T}{T_c}},\end{aligned}$$

where we have used $(T/T_c)^2 \approx 1$.



Grading scheme for Task C.6.	**Pts**
Using the proper approximation for tanh	0.1
Correct non-trivial solutions for $\bar{s}$, even if not fully simplified (0.1 each)	0.2
Sketch $\bar{s} = 0$ as the only solution above T_c	0.1
The non trivial solutions are vertical at T_c	0.2
$\bar{s} = 0$ is sketched as a solution for $T < T_c$	0.1
The two non-trivial solutions monotonically increase to ± 1 at $T = 0$ (0.1 each)	0.2
Either non-trivial solutions have a zero slope approaching $T = 0$.	0.1
Total	**1.0**

Grading note 1:The vertical slope at T_c has to be very clear. It is an important physical signature of phase transitions, so the student has to emphasize it in the sketch to receive credit for it.

Grading note 2: the credit associated with slope approaching zero as $T \to 0$ can be given even if the student does not plot the negative solution.

C.7: In the absence of magnetic fields, when $T > T_c$, there is only one solution at $\bar{s} = 0$, which means that the system cannot maintain a net polarization. However, applying a magnetic field leads to a net magnetization, which is a characteristic of **paramagnets**. On the other hand, when $T < T_c$, the system can support a net magnetization even in the absence of magnetic fields, which is the characteristic of **ferromagnets**.

Grading scheme for Task C.7.	**Pts**
Correct choice for $T > T_c$	0.1
Correct choice for $T < T_c$	0.1
Total	**0.2**

Grading note: No justification is needed for credit. If multiple choices are chosen, then no credit. In case the choices were changed, only give credit if the final choice is correct and clear.