Baltic Way 1992

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Problems and solutions

1. Let p and q be two consecutive odd prime numbers. Prove that $p+q$ is a product of at least three positive integers greater than 1 (not necessarily different).

   Solution. Since $q-p=2k$ is even, we have $p+q=2(p+k)$. It is clear that $p<p+k<p+2k=q$. Therefore $p+k$ is not prime and, consequently, is a product of two positive integers greater than 1.

2. Denote by $d(n)$ the number of all positive divisors of a positive integer n (including 1 and n). Prove that there are infinitely many n such that $\frac{n}{d(n)}$ is an integer.

   Solution. Consider numbers of the form p^{p^n-1} where p is an arbitrary prime number and $n=1, 2, \ldots$.

3. Find an infinite non-constant arithmetic progression of positive integers such that each term is neither a sum of two squares, nor a sum of two cubes (of positive integers).

   Solution. For any natural number n, we have $n^2\equiv 0$ or $n^2\equiv 1 \pmod 4$ and $n^3\equiv 0$ or $n^3\equiv \pm 1 \pmod 9$. Thus $\{36n+3 \mid n=1,2,\ldots\}$ is a progression with the required property.

4. Is it possible to draw a hexagon with vertices in the knots of an integer lattice so that the squares of the lengths of the sides are six consecutive positive integers?

   Solution. The sum of any six consecutive positive integers is odd. On the other hand, the sum of the squares of the lengths of the sides of the hexagon is equal to the sum of the squares of their projections onto the two axes. But this number has the same parity as the sum of the projections themselves, the latter being obviously even.

5. Given that $a^2+b^2+(a+b)^2=c^2+d^2+(c+d)^2$, prove that $a^4+b^4+(a+b)^4=c^4+d^4+(c+d)^4$.

   Solution. Use the identity $\big(a^2+b^2+(a+b)^2\big)^2=2\big(a^4+b^4+(a+b)^4\big)$.

6. Prove that the product of the 99 numbers of the form $\frac{k^3-1}{k^3+1}$ where $k=2, 3, \ldots, 100$, is greater than $\frac{2}{3}$.

   Solution. Note that

$$\frac{k^3-1}{k^3+1}=\frac{(k-1)(k^2+k+1)}{(k+1)(k^2-k+1)}=\frac{(k-1)(k^2+k+1)}{(k+1)\big((k-1)^2+(k-1)+1\big)}.$$

   After obvious cancellations we get

$$\prod_{k=2}^{100}\frac{k^3-1}{k^3+1}=\frac{1\cdot 2\cdot(100^2+100+1)}{100\cdot 101\cdot(1^2+1+1)}>\frac{2}{3}.$$

7. Let $a=\sqrt[1992]{1992}$. Which number is greater:

$$\left.a^{a^{a^{\cdot^{\cdot^{\cdot^{a}}}}}}\right\}1992$$

   or 1992?

   Solution. The first of these numbers is less than

$$\left.a^{a^{a^{\cdot^{\cdot^{\cdot^{1992}}}}}}\right\}1992=\left.a^{a^{a^{\cdot^{\cdot^{\cdot^{1992}}}}}}\right\}1991=\ldots=1992.$$

8. Find all integers satisfying the equation $2^x\cdot(4-x)=2x+4$.

   Solution. Since 2^x must be positive, we have $\frac{2x+4}{4-x}>0$ yielding $-2<x<4$. Thus it suffices to check the points $-1, 0, 1, 2, 3$. The three solutions are $x=0, 1, 2$.

9. A polynomial $f(x) = x^3 + ax^2 + bx + c$ is such that $b < 0$ and $ab = 9c$. Prove that the polynomial has three different real roots.

Solution. Consider the derivative $f'(x) = 3x^2 + 2ax + b$. Since $b < 0$, it has two real roots x_1 and x_2. Since $f(x) \to \pm\infty$ as $x \to \pm\infty$, it is sufficient to check that $f(x_1)$ and $f(x_2)$ have different signs, i.e., $f(x_1)f(x_2) < 0$. Dividing $f(x)$ by $f'(x)$ and using the equality $ab = 9c$ we find that the remainder is equal to $x(\frac{2}{3}b - \frac{2}{9}a^2)$. Now, as $x_1x_2 = \frac{b}{3} < 0$ we have $f(x_1)f(x_2) = x_1x_2(\frac{2}{3}b - \frac{2}{9}a^2)^2 < 0$.

10. Find all fourth degree polynomials $p(x)$ such that the following four conditions are satisfied:

   (i) $p(x) = p(-x)$ for all x.
   (ii) $p(x) \geq 0$ for all x.
   (iii) $p(0) = 1$.
   (iv) $p(x)$ has exactly two local minimum points x_1 and x_2 such that $|x_1 - x_2| = 2$.

Solution. Let $p(x) = ax^4 + bx^3 + cx^2 + dx + e$ with $a \neq 0$. From (i)–(iii) we get $b = d = 0$, $a > 0$ and $e = 1$. From (iv) it follows that $p'(x) = 4ax^3 + 2cx$ has at least two different real roots. Since $a > 0$, we have $c < 0$ and $p'(x)$ has three roots $x = 0$, $x = \pm\sqrt{-c/(2a)}$. The minimum points mentioned in (iv) must be $x = \pm\sqrt{-c/(2a)}$, so $2\sqrt{-c/(2a)} = 2$ and $c = -2a$. Finally, by (ii) we have $p(x) = a(x^2 - 1)^2 + 1 - a \geq 0$ for all x, which implies $0 < a \leq 1$. It is easy to check that every such polynomial satisfies the conditions (i)–(iv).

11. Let $\mathbb{Q}^+$ denote the set of positive rational numbers. Show that there exists one and only one function $f : \mathbb{Q}^+ \to \mathbb{Q}^+$ satisfying the following conditions:

   (i) If $0 < q < \frac{1}{2}$ then $f(q) = 1 + f\left(\frac{q}{1-2q}\right)$.
   (ii) If $1 < q \leq 2$ then $f(q) = 1 + f(q - 1)$.
   (iii) $f(q) \cdot f\left(\frac{1}{q}\right) = 1$ for all $q \in \mathbb{Q}^+$.

Solution. By condition (iii) we have $f(1) = 1$. Applying condition (iii) to each of (i) and (ii) gives two new conditions (i') and (ii') taking care of $q > 2$ and $\frac{1}{2} \leq q < 1$ respectively. Now, for any rational number $\frac{a}{b} \neq 1$ we can use (i), (i'), (ii) or (ii') to express $f\left(\frac{a}{b}\right)$ in terms of $f\left(\frac{a'}{b'}\right)$ where $a' + b' < a + b$. The recursion therefore finishes in a finite number of steps, when we can use $f(1) = 1$. Thus we have established that such a function f exists, and is uniquely determined by the given conditions.

Remark. Initially it was also required to determine all fixed points of the function f, i.e., all solutions q of the equation $f(q) = q$, but the Jury of the contest decided to simplify the problem. Here we present a solution. First note that if q is a fixed point, then so is $\frac{1}{q}$. By (i), if $0 < q < \frac{1}{2}$ is a fixed point, then $f\left(\frac{q}{1-2q}\right) = q - 1 < 0$ which is impossible, so there are no fixed points $0 < q < \frac{1}{2}$ or $q > 2$. Now, for a fixed point $1 < \frac{a}{b} \leq 2$ (ii) easily gives us that $\frac{a}{b} - 1 = \frac{a-b}{b}$ and $\frac{b}{a-b}$ are fixed points too. It is easy to see that $1 \leq \frac{b}{a-b} \leq 2$ (the latter holds because $\frac{b}{a-b}$ is a fixed point). As the sum of the numerator and denominator of the new fixed point is strictly less than $a + b$ we can continue in this manner until, in a finite number of steps, we arrive at the fixed point 1. By reversing the process, any fixed point $q > 1$ can be constructed by repeatedly using the condition that if $\frac{a}{b} > 1$ is a fixed point then so is $\frac{a+b}{a}$, starting with $a = b = 1$. It is now an easy exercise to see that these fixed points have the form $\frac{F_{n+1}}{F_n}$ where $\{F_n\}_{n\in\mathbb{N}}$ is the sequence of Fibonacci numbers.

12. Let $\mathbb{N}$ denote the set of positive integers. Let $\varphi : \mathbb{N} \to \mathbb{N}$ be a *bijective* function and assume that there exists a finite limit

$$\lim_{n\to\infty} \frac{\varphi(n)}{n} = L.$$

What are the possible values of L?

Solution. In this solution we allow L to be ∞ as well. We show that $L = 1$ is the only possible value. Assume that $L > 1$. Then there exists a number N such that for any $n \geq N$ we have $\frac{\varphi(n)}{n} > 1$ and thus $\varphi(n) \geq n + 1 \geq N + 1$. But then φ cannot be bijective, since the numbers $1, 2, \ldots, N - 1$ cannot be bijectively mapped onto $1, 2, \ldots, N$.

Now assume that $L < 1$. Since φ is bijective we clearly have $\varphi(n) \to \infty$ as $n \to \infty$. Then

$$\lim_{n\to\infty} \frac{\varphi^{-1}(n)}{n} = \lim_{n\to\infty} \frac{\varphi^{-1}(\varphi(n))}{\varphi(n)} = \lim_{n\to\infty} \frac{n}{\varphi(n)} = \frac{1}{L} > 1,$$

i.e., $\lim\limits_{n\to\infty} \frac{\varphi^{-1}(n)}{n} > 1$, which is a contradiction since φ^{-1} is also bijective. (When $L = 0$ we interpret $\frac{1}{L}$ as ∞).

13. Prove that for any positive $x_1, x_2, \ldots, x_n$ and $y_1, y_2, \ldots, y_n$ the inequality

$$\sum_{i=1}^{n} \frac{1}{x_i y_i} \geq \frac{4n^2}{\sum\limits_{i=1}^{n} (x_i + y_i)^2}$$

holds.

Solution. Since $(x_i + y_i)^2 \geq 4x_i y_i$, it is sufficient to prove that

$$\Big(\sum_{i=1}^{n} \frac{1}{x_i y_i}\Big)\Big(\sum_{i=1}^{n} x_i y_i\Big) \geq n^2.$$

This can easily be done by induction using the fact that $a + \frac{1}{a} \geq 2$ for any $a > 0$. It also follows directly from the Cauchy-Schwarz inequality.

14. There is a finite number of towns in a country. They are connected by one direction roads. It is known that, for any two towns, one of them can be reached from the other one. Prove that there is a town such that all the remaining towns can be reached from it.

Solution. Consider a town A from which a maximal number of towns can be reached. Suppose there is a town B which cannot be reached from A. Then A can be reached from B and so one can reach more towns from B than from A, a contradiction.

15. Noah has to fit 8 species of animals into 4 cages of the ark. He plans to put species in each cage. It turns out that, for each species, there are at most 3 other species with which it cannot share the accommodation. Prove that there is a way to assign the animals to their cages so that each species shares a cage with compatible species.

Solution. Start assigning the species to cages in an arbitrary order. Since for each species there are at most three species incompatible with it, we can always add it to one of the four cages.

Remark. Initially the problem was posed as follows: "…He plans to put *two* species in each cage…" Because of a misprint the word "two" disappeared, and the problem became trivial. We give a solution to the original problem. Start with the distribution obtained above. If in some cage A there are more than three species, then there is also a cage B with at most one species and this species is compatible with at least one species in cage A, which we can then transfer to cage B. Thus we may assume that there are at most three species in each cage. If there are two cages with 3 species, then we can obviously transfer one of these 6 species to one of the remaining two cages. Now, assume the four cages contain 1, 2, 2 and 3 species respectively. If the species in the first cage is compatible with one in the fourth cage, we can transfer that species to the first cage, and we are done. Otherwise, for an arbitrary species X in the fourth cage there exists a species compatible with it in either the second or the third cage. Transfer the other species from that cage to the first cage, and then X to that cage.

16. All faces of a convex polyhedron are parallelograms. Can the polyhedron have exactly 1992 faces?

Solution. No, it cannot. Let us call a series of faces $F_1, F_2, \ldots, F_k$ a *ring* if the pairs $(F_1, F_2), (F_2, F_3), \ldots, (F_{k-1}, F_k), (F_k, F_1)$ each have a common edge and all these common edges are parallel. It is not difficult to see that any two rings have exactly two common faces and, conversely, each face belongs to exactly two rings. Therefore, if there are n rings then the total number of faces must be $2\binom{n}{2} = n(n-1)$. But there is no positive integer n such that $n(n-1) = 1992$.

Remark. The above solution, which is the only one proposed that is known to us, is not correct. For a counterexample, consider a cube with side 2 built up of four unit cubes, and take the polyhedron with 24 faces built up of the faces of the unit cubes that face the outside. This polyhedron has rings that do not have any faces in common. Moreover, by subdividing faces into rectangles sufficiently many times, we can obtain a polyhedron with 1992 faces.

17. Quadrangle $ABCD$ is inscribed in a circle with radius 1 in such a way that one diagonal, AC, is a diameter of the circle, while the other diagonal, BD, is as long as AB. The diagonals intersect in P. It is known that the length of PC is $\frac{2}{5}$. How long is the side CD?

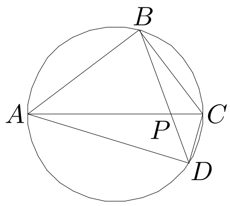

Figure 1

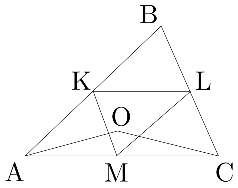

Figure 2

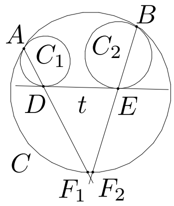

Figure 3

Solution. Let $\angle ACD = 2\alpha$ (see Figure 1). Then $\angle CAD = \frac{\pi}{2} - 2\alpha$, $\angle ABD = 2\alpha$, $\angle ADB = \frac{\pi}{2} - \alpha$ and $\angle CDB = \alpha$. The sine theorem applied to triangles DCP and DAP yields

$$\frac{|DP|}{\sin 2\alpha} = \frac{2}{5\sin\alpha}$$

and

$$\frac{|DP|}{\sin\left(\frac{\pi}{2} - 2\alpha\right)} = \frac{8}{5\sin\left(\frac{\pi}{2} - \alpha\right)}.$$

Combining these equalities we have

$$\frac{2\sin 2\alpha}{5\sin\alpha} = \frac{8\cos 2\alpha}{5\cos\alpha},$$

which gives $4\sin\alpha\cos^2\alpha = 8\cos 2\alpha\sin\alpha$ and $\cos 2\alpha + 1 = 4\cos 2\alpha$. So we get $\cos 2\alpha = \frac{1}{3}$ and $|CD| = 2\cos 2\alpha = \frac{2}{3}$.

18. Show that in a non-obtuse triangle the perimeter of the triangle is always greater than two times the diameter of the circumcircle.

    Solution. Let K, L, M be the midpoints of the sides AB, BC, AC of a non-obtuse triangle ABC (see Figure 2). Note that the centre O of the circumcircle is inside the triangle KLM (or at one of its vertices if ABC is a right-angled triangle). Therefore $|AK|+|KL|+|LC| > |AO|+|OC|$ and hence $|AB|+|AC|+|BC| > 2(|AO| + |OC|) = 2d$, where d is the diameter of the circumcircle.

19. Let C be a circle in the plane. Let C_1 and C_2 be non-intersecting circles touching C internally at points A and B respectively. Let t be a common tangent of C_1 and C_2, touching them at points D and E respectively, such that both C_1 and C_2 are on the same side of t. Let F be the point of intersection of AD and BE. Show that F lies on C.

    Solution. Let F_1 be the second intersection point of the line AD and the circle C (see Figure 3). Consider the homothety with centre A which maps D onto F_1. This homothety maps the circle C_1 onto C and the tangent line t of C_1 onto the tangent line of the circle C at F_1. Let us do the same with the circle C_2 and the line BE: let F_2 be their intersection point and consider the homothety with centre B, mapping E onto F_2, C_2 onto C and t onto the tangent of C at point F_2. Since the tangents of C at F_1 and F_2 are both parallel to t, they must coincide, and so must the points F_1 and F_2.

20. Let $a \le b \le c$ be the sides of a right triangle, and let $2p$ be its perimeter. Show that

    $$p(p-c) = (p-a)(p-b) = S,$$

    where S is the area of the triangle.

    Solution. By straightforward computation, we find:

    $$p(p-c) = \frac{1}{4}\big((a+b)^2 - c^2\big) = \frac{ab}{2} = S,$$
    $$(p-a)(p-b) = \frac{1}{4}\big(c^2 - (a-b)^2\big) = \frac{ab}{2} = S.$$