

Solution - Oscillating rope

A) It is evident from the figure that the curvature of the rope in the fundamental vibration is very small. It infers for a possibility to model the fundamental vibration as a swinging of a rigid uniform rod of length L about a pivot point at its end. The moment of inertia of the rod is:

$$I = \frac{mL^2}{3}$$

and the distance from the center-of-mass to the pivot point is:

$$b = L/2$$

Therefore, the frequency of the fundamental vibration is approximated as:

$$f_1 = \frac{1}{2\pi} \sqrt{\frac{mgb}{I}} = \frac{1}{2\pi} \sqrt{\frac{2L}{3g}} \approx 0.61 \text{ Hz}$$

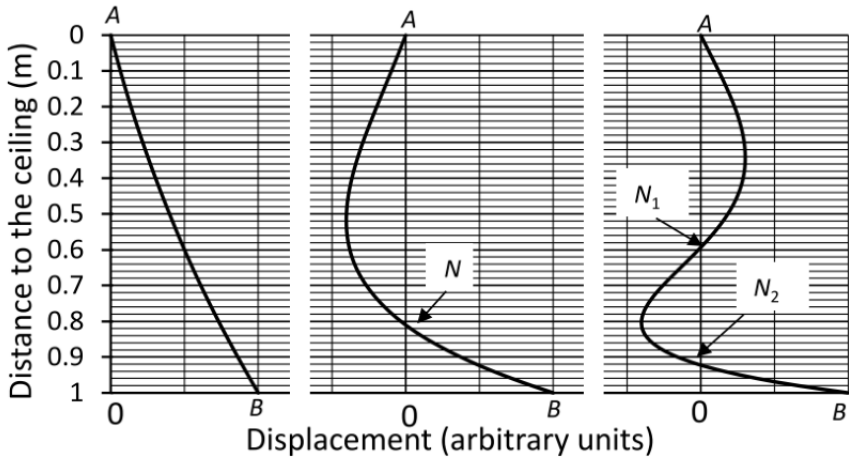
Correspondingly, the period of the fundamental vibration is:

$$T_1 = \frac{1}{2\pi} \sqrt{\frac{I}{mgb}} = \frac{1}{2\pi} \sqrt{\frac{3g}{2L}} \approx 1.6 \text{ s}$$

B) Whatever model for estimating of f_1 is being used, one may deduce on the basis of dimensionality arguments that the k -th natural frequency of the rope is:

$$f_k = c_k \sqrt{\frac{g}{L}}$$

where c_k is a dimensionless numeric coefficient depending on the consecutive mode number k only. Let A and B be the suspension point and the free end of the rope respectively, and N be the node on the rope for the second natural vibration (see the figure).



Since the node point is at rest (in the small-amplitude approximation), the vibration of the part NB could be considered as a fundamental vibration of a rope of length LNA about a suspension point N . Therefore:

$$f_2(L) \equiv f_1(L - NA)$$

Hence one may write:

$$\frac{f_2(L)}{f_1(L)} = \frac{f_1(L - NA)}{f_1(L)} = \sqrt{\frac{L}{L - NA}}$$

Since the absolute displacement is much smaller than the length of the rope, the distances could be measured in a vertical direction, to the ceiling, instead along the rope. Therefore, by taking $L = 1 \text{ m}$, and $NA \approx 0.8 \text{ m}$, we obtain:

$$\frac{f_2}{f_1} \approx 2.2$$

Similarly, the vibration of the part N_1B in the third eigenmode is equivalent to the second natural vibration of a rope of length $L - N_1A \approx 0.4\text{ m}$. In analogy to the first case:

$$f_3(L) \equiv f_2(L - NA)$$

and

$$\frac{f_3(L)}{f_2(L)} = \frac{f_2(L - N_1A)}{f_2(L)} = \sqrt{\frac{L}{L - N_1A}} \approx 1.6$$

Therefore:

$$\frac{f_3}{f_1} = \frac{f_2}{f_1} \times \frac{f_3}{f_2} \approx 3.5$$

Finally:

$$f_1 : f_2 : f_3 \approx 1 : 2.2 : 3.5$$

Part	Marking scheme	Points
A	States explicitly or realizes (proper drawing or notations) the physical pendulum analogy	1.0
	Correct expression for the moment of inertia	1.0
	Determination of the position of the center of mass	0.5
	Correct formula for the period/frequency of a physical pendulum	1.0
	Calculates $f_1 = 0.61\text{ Hz}$ with two significant digits	0.5
	Subtotal on A	4.0
B	States or realizes (proper graph or formula) that the vibration of the rope below a node point is similar to a lower order vibration of a shorter rope.	1.0
	Uses dimensionality arguments to argue that $f_k = c_k(g/L)^{1/2}$	1.0
	Applies similarity arguments to f_1 and f_2 and derives $f_2/f_1 = (L/L-NA)^{1/2}$	1.0
	Reads correctly NA from the graph	0.5
	Calculates $f_2/f_1 = 2.2$ to a precision of 2 significant digits	0.5
	Applies similarity arguments to f_3 and f_2 (or f_1) and derives $f_3/f_2 = (L/L-N_1A)^{1/2}$ or $f_3/f_1 = (L/L-N_2A)^{1/2}$	1.0
	Reads correctly N_1A (N_2A) from the graph.	0.5
	Calculates $f_3/f_1 = 3.5$ to a precision of 2 significant digits.	0.5
	Subtotal on B	6.0