2 Black box

Let the tension forces in the two springs be F_1 and F_2, respectively. Let the height of the ceiling of the box be y_1 and let the heights of the masses be y_2 and y_3 (we will assume for simplicity that the masses have zero height). Let a_1, a_2, a_3 be the respective accelerations. We get the following equations of motion (ignoring all drag forces):

$$m_1 a_1 = F - F_1 - m_1 g$$

$$m_2 a_2 = F_1 - F_2 - m_2 g$$

$$m_3 a_3 = F_2 - m_3 g$$

Since the springs are nonlinear, $F_1 \neq k_1(y_1 - y_2)$ and $F_2 \neq k_2(y_2 - y_3)$ in general, but we know that for small displacements near equilibrium $k_1 = \frac{\Delta F_1}{\Delta(y_1 - y_2)}$ and $k_2 = \frac{\Delta F_2}{\Delta(y_2 - y_3)}$.

2.1 Finding $m_1 + m_2 + m_3$

When the system is at rest and at equilibrium, then the force needed to hold the box is the total gravitational force $F_0 = (m_1 + m_2 + m_3)g$ (we can get the same result if we plug in $a_1 = a_2 = a_3 = 0$ to the equations of motion).

To measure F_0, we find the value of F when the box is at rest ($a_1 = 0$). We notice that the force is constant which means that the system is initially already in equilibrium.

After averaging 10 first values we get $F_0 \approx 14.774\,\mathrm{N}$ and

$$m_1 + m_2 + m_3 = \frac{F_0}{g} = \frac{14.774\,\mathrm{N}}{9.81\,\mathrm{N/kg}} \approx 1.506\,\mathrm{kg}.$$

Exact answer: 1.506 kg.

	Finding $m_1 + m_2 + m_3$	
1a	Notice that $F = g\sum m_i$ when $a_1 = 0$	0.5
1b	Measurement for F_0 ($14.77 \pm 0.10\,\mathrm{N}$) 0 points for only having a measurement without an idea how to use it	0.3
1c	$\sum m_i$ in range $1.51 \pm 0.01\,\mathrm{kg}$	0.2
	Total:	**1.0**

Note: Measurement for F_0 is needed for full points, even if $\sum m_i$ is correct. Solutions with raw data missing get 0.7 points. Solutions using $\frac{F_0}{g}$ implicitly as the sum of masses get 0.2 points from 1c.

2.2 Finding m_1

We get from the first equation of motion that $F = m_1 a_1 + m_1 g + F_1$. The spring force F_1 depends only on the positions (and is the same at the beginning of every experiment), so the force F at the beginning of the experiment depends only on the acceleration.

Therefore, we can measure how much the initial force changes with acceleration to get m_1. We will use maximum acceleration ($30\,\mathrm{m/s^2}$) for highest accuracy. The average of three values is $F_{30} \approx 40.487\,\mathrm{N}$, so

$$m_1 = \frac{F_{30} - F_0}{a} = \frac{(40.487 - 14.774)\,\mathrm{N}}{30\,\mathrm{m/s^2}} \approx 0.857\,\mathrm{kg}.$$

We also conclude that $m_2 + m_3 = 0.649\,\mathrm{kg}$.

(To even increase accuracy, one could compare F_{30} and F_{-30} and find their difference.)

Exact answer: 0.857 kg.

Finding m_1		
2a	$F = m_1 a_1 + m_1 g + F_1$ or any equivalent equation of motion (max points even if F_1 has been incorrectly substituted with $k_1(y_1 - y_2)$)	0.5
2b	Idea that $m_1 = \frac{\Delta F}{\Delta a_1}$	0.5
2c	Using $\Delta a_1 \geq 10\,\mathrm{m/s^2}$	0.2
2d	m_1 in range $0.857 \pm 0.002\,\mathrm{kg}$	0.8
	m_1 in range $0.857 \pm 0.010\,\mathrm{kg}$	0.6
	m_1 in range $0.857 \pm 0.050\,\mathrm{kg}$	0.3
	Total:	**2.0**

Note: Using free-fall ($a_1 = -g$) without repeated measurements gets 0 points from 2c. Full points are given if $\Delta a_1 < 10\,\mathrm{m/s^2}$ but several measurements are used that give at least as good accuracy overall.

2.3 Finding k_1

2.3.1 Method 1: Change of force after a fast movement of the box

We will quickly accelerate and then decelerate the box (to avoid drag forces). When we change the height of the box quickly and the time is short enough, we can assume that the second mass stays approximately at rest.

(Formally, if $\Delta y_1 = \frac{a_1}{2} t^2$, then $m_2 a_2 = k_1 \Delta y_1 - k_1 \Delta y_2 - \Delta F_2 \leq k_1 \Delta y_1$, therefore $a_2 \leq \frac{k_1}{2m_2} t^2 \cdot a_1$. The assumption holds if $\frac{k_1}{2m_2} t^2 \ll 1$.)

Therefore, if we accelerate the box with acceleration a_1 for time t and then with $-a_1$ for time t, then $\Delta F \approx k_1 \Delta y_1 = k_1 a_1 t^2$.

To have the best accuracy we will do two experiments with $a_1 = 30\,\mathrm{m/s^2}$ and $a_1 = -30\,\mathrm{m/s^2}$, respectively. We will use $t = 0.01\,\mathrm{s}$ (smallest time possible). We will also repeat each experiment 5 times. After averaging the results, we get the forces at $2t = 0.02\,\mathrm{s}$ to be $F_\mathrm{u} \approx 14.890\,\mathrm{N}$ and $F_\mathrm{d} \approx 14.652\,\mathrm{N}$. Therefore

$$k_1 \approx \frac{F_\mathrm{u} - F_\mathrm{d}}{2 a_1 t^2} = \frac{(14.890 - 14.652)\,\mathrm{N}}{2 \cdot 30\,\mathrm{m/s^2} \cdot (0.01\,\mathrm{s})^2} \approx 39.7\,\mathrm{N/m}$$

Exact answer: 39.2 N/m.

Finding k_1, Method 1		
3.1a	Idea to use method	0.5
3.1b	Notice that if t is small, then $\Delta y_1 \gg \Delta y_2$	0.5
3.1c	Correct formula for k_1	0.5
3.1d	At least 3 measurements	0.1
3.1e	$\Delta a_1 \geq 30\,\mathrm{m/s^2}$	0.2
3.1f	$2t \leq 0.08\,\mathrm{s}$	0.2
3.1g	k_1 in range $39.2 \pm 1.0\,\mathrm{N/m}$ k_1 in range $39 \pm 4\,\mathrm{N/m}$ k_1 in range $39 \pm 8\,\mathrm{N/m}$ k_1 in range $39 \pm 15\,\mathrm{N/m}$	1.0 0.7 0.4 0.2
	Total:	**3.0**

2.3.2 Method 2: Change of force while accelerating the box

We will accelerate the box with constant acceleration a_1 and similarly as in the previous method conclude that $\Delta F \approx \frac{k_1 a_1}{2} t^2$ when t is small. This method is, however, less accurate than the previous method because drag force is nonnegligible for large values of a_1 but the resolution of ΔF is small for small values of a_1.

Choosing, for example, $a_1 = 30\,\mathrm{m/s^2}$, $t = 0.02\,\mathrm{s}$ and averaging 5 values gives $F_{t=0} \approx 40.482\,\mathrm{N}$ and $F_{t=0.02} \approx 40.792\,\mathrm{N}$ and

$$k_1 \approx \frac{2\Delta F}{a_1 t^2} = \frac{2 \cdot (40.792 - 40.482)\,\mathrm{N}}{30\,\mathrm{m/s^2} \cdot (0.02\,\mathrm{s})^2} \approx 51.7\,\mathrm{N/m}$$

The answer is $\sim 30\%$ larger than the correct answer because the drag force of the box at $t = 0.02\,\mathrm{s}$ is approximately 0.08 N.

A better choice would be $a_1 = 5\,\mathrm{m/s^2}$ and $t = 0.02\,\mathrm{s}$. Averaging 5 values gives $F_{t=0} \approx 19.058\,\mathrm{N}$ and $F_{t=0.02} \approx 19.100\,\mathrm{N}$ and

$$k_1 \approx \frac{2\Delta F}{a_1 t^2} = \frac{2 \cdot (19.100 - 19.058)\,\mathrm{N}}{5\,\mathrm{m/s^2} \cdot (0.02\,\mathrm{s})^2} \approx 42.0\,\mathrm{N/m}$$

The drag force has a much smaller effect (approximately 0.002 N).

Finding k_1, Method 2		
3.2a	Idea to use method	0.5
3.2b	Notice that if t is small, then $\Delta y_1 \gg \Delta y_2$	0.5
3.2c	Correct formula for k_1	0.5
3.2d	At least 3 measurements	0.1
3.2e	$2\,\mathrm{m/s^2} \leq a_1 \leq 10\,\mathrm{m/s^2}$	0.2
3.2f	$t \leq 0.08\,\mathrm{s}$	0.2
3.1g	k_1 in range $39.2 \pm 1.0\,\mathrm{N/m}$ k_1 in range $39 \pm 4\,\mathrm{N/m}$ k_1 in range $39 \pm 8\,\mathrm{N/m}$ k_1 in range $39 \pm 15\,\mathrm{N/m}$	1.0 0.7 0.4 0.2
	Total:	**3.0**

Note: A correct answer without any justification or obtained with a physically nonsensible method gives 0 points.

2.3.3 Method 3: Estimating $y_1 - y_2$ at equilibrium and using $F_1 \approx k_1(y_1 - y_2)$

Although the springs are nonlinear, we can estimate k_1 by $k_1 \approx \frac{F_1}{y_1 - y_2}$ which would be true if the springs were perfectly linear.

At equilibrium

$$F_1 = F_0 - m_1 g \approx 14.774\,\mathrm{N} - 0.857 \cdot 9.81\,\mathrm{N} \approx 6.367\,\mathrm{N}$$

If we accelerate the box quickly downwards, then by measuring the time t for the box to collide with mass 2, we can estimate the initial value of $y_1 - y_2$ by $\Delta y_1 = \frac{a_1}{2} t^2$.

Using binary search we can find that $t \leq 0.13\,\mathrm{s}$ if $|a_1| \geq 26.7\,\mathrm{m/s^2}$ and $t \geq 0.13\,\mathrm{s}$ if $|a_2| \leq 26.6\,\mathrm{m/s^2}$.

Therefore

$$y_1 - y_2 \approx \frac{26.7\,\mathrm{m/s^2}}{2} \cdot (0.13\,\mathrm{s})^2 \approx 0.226\,\mathrm{m}$$

and

$$k_1 \approx \frac{F_1}{y_1 - y_2} = \frac{6.367\,\mathrm{N}}{0.226\,\mathrm{m}} \approx 28.2\,\mathrm{N/m}$$

This method underestimates the value both due to nonlinearity of springs and because it overestimates $y_1 - y_2$ (the actual value is 0.179 m).

Finding k_1, Method 3		
3.3a	Idea to use method	0.5
3.3b	Correctly estimate $y_1 - y_2$	0.5
3.3c	Correct formula for k_1	0.5
	Total:	**1.5**

Note: This method is worth 1.5 points since it is very inaccurate. Estimating the distance $y_1 - y_2$ without an idea how to use it gives 0 points.

Method for eye-balling k_1 from slow normal mode frequency assuming a rigid connection between m_2 and m_3 was rewarded with 0.5+0.5 points for idea and formula if significant progress was made (a reasonable value for the normal mode period and k_1 or $\frac{k_1}{m_2+m_3}$ was found). Simply stating $T = 2\pi\sqrt{\frac{m_2+m_3}{k_1}}$ gave 0 points.

2.4 Finding m_2, m_3 and k_2

2.4.1 Method 1: Finding natural frequencies

This method is very accurate, but needs a lot algebraic manipulation to solve for two parameters. This method could also be used to find one parameter if the other has been already found using alternative methods.

At first we will find the natural frequencies of the system when the box is at rest. Let $x_2 = \Delta y_2$ and $x_3 = \Delta y_3$ be small displacements near equilibrium. Then

$$m_2 \ddot{x}_2 = -k_1 x_2 - k_2(x_2 - x_3)$$

$$m_3 \ddot{x}_3 = k_2(x_2 - x_3)$$

The equations can be solved by taking $x_2 = A\cos(\omega t)$ and $x_3 = B\cos(\omega t)$, where A and B are constants.

(Alternatively, one can use complex numbers: $\tilde{x}_2 = Ae^{i\omega t}$ and $\tilde{x}_3 = Be^{i\omega t}$.)

We see that $\ddot{x}_2 = -\omega^2 A\cos(\omega t)$ and $\ddot{x}_3 = -\omega^2 B\cos(\omega t)$, hence

$$-m_2\omega^2 A\cos(\omega t) = -k_1 A\cos(\omega t) - k_2(A-B)\cos(\omega t)$$

$$-m_3\omega^2 B\cos(\omega t) = k_2(A-B)\cos(\omega t)$$

We see that the time dependence cancels out

$$-m_2\omega^2 A = -k_1 A - k_2 A + k_2 B$$

$$-m_3\omega^2 B = k_2 A - k_2 B$$

We get from the second equation that $B = \frac{k_2 A}{k_2 - m_3\omega^2}$, so after substituting to the first equation we get

$$-m_2\omega^2 A = -k_1 A - k_2 A + \frac{k_2^2}{k_2 - m_3\omega^2}A$$

As expected, A cancels out (because natural frequency does not depend on the amplitude of the oscillations) and we get

$$-m_2\omega^2(k_2 - m_3\omega^2) + (k_1+k_2)(k_2 - m_3\omega^2) - k_2^2 = 0$$

$$m_2 m_3\omega^4 - k_2 m_2\omega^2 - (k_1+k_2)m_3\omega^2 + k_1 k_2 = 0$$

$$\omega^4 - \left(\frac{k_2}{m_3} + \frac{k_1+k_2}{m_2}\right)\omega^2 + \frac{k_1 k_2}{m_2 m_3} = 0$$

The solutions to this biquadratic equation are the natural angular frequencies. If we know the solutions ω_1 and ω_2, we know from the Vieta's formulas that

$$\frac{k_2}{m_3} + \frac{k_1+k_2}{m_2} = c_1$$

$$\frac{k_1 k_2}{m_2 m_3} = c_2,$$

where $c_1 = \omega_1^2 + \omega_2^2$ and $c_2 = \omega_1^2\omega_2^2$.

We find that

$$\frac{m_2}{k_1} + \frac{m_3}{k_2} + \frac{m_3}{k_1} = \frac{c_1}{c_2}$$

$$\frac{m_3}{k_2} = \frac{c_1}{c_2} - \frac{m_2+m_3}{k_1}$$

$$\frac{k_1}{m_2 c_2} = \frac{c_1}{c_2} - \frac{m_2+m_3}{k_1}$$

$$m_2 = \frac{k_1^2}{c_1 k_1 - c_2(m_2+m_3)}$$

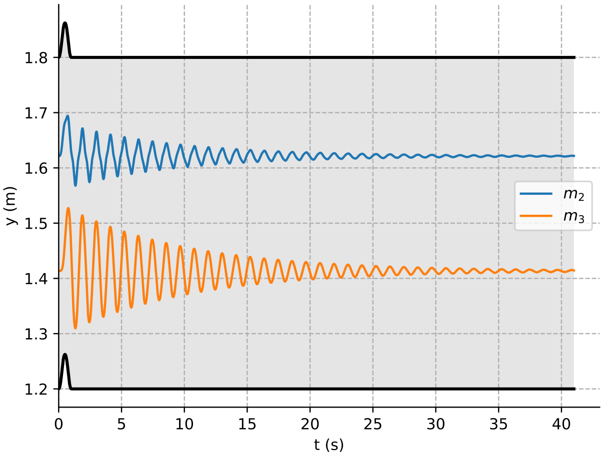
This equation allows us to find m_2. After this, it is easy to also find m_3 and k_2.

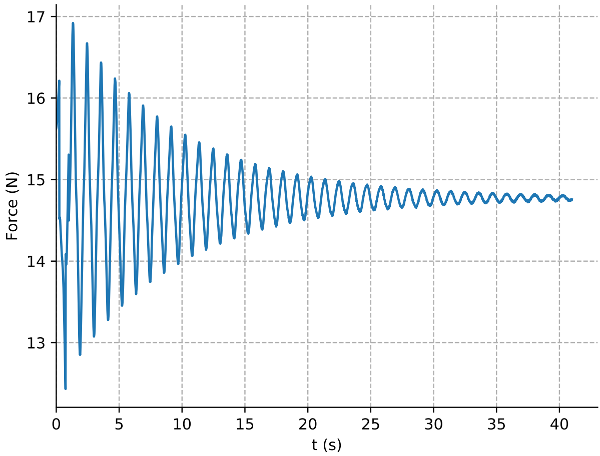
To find the natural frequencies, one can oscillate the box with different frequencies, stop oscillating and look at how force changes in time. Using trial and error we can get two estimates $T_1 \approx 1$ s and $T_2 \approx 0.4$ s.

To find the smaller frequency, we can, for example, give the box a pulse with 1 s duration.

We want to be sure that the amplitude of the oscillations is small enough when we measure the period (to avoid nonlinearity of springs). We find

$$T_1 = \frac{(34.70 - 20.27)\,\text{s}}{13} \approx 1.11\,\text{s}.$$





Similarly, we can amplify the larger natural frequency by oscillating the box or giving a shorter pulse.

We find

$$T_2 = \frac{(20.00 - 9.94)\,\mathrm{s}}{27} \approx 0.373\,\mathrm{s}.$$

Therefore

$$\omega_1^2 = \left(\frac{2\pi}{T_1}\right)^2 \approx 32.04\,\mathrm{Hz}^2$$

$$\omega_2^2 = \left(\frac{2\pi}{T_1}\right)^2 \approx 283.8\,\mathrm{Hz}^2$$

We can then find m_2 by calculating c_1 and c_2:

$$c_1 = \omega_1^2 + \omega_2^2 \approx 315.8\,\mathrm{Hz}^2$$

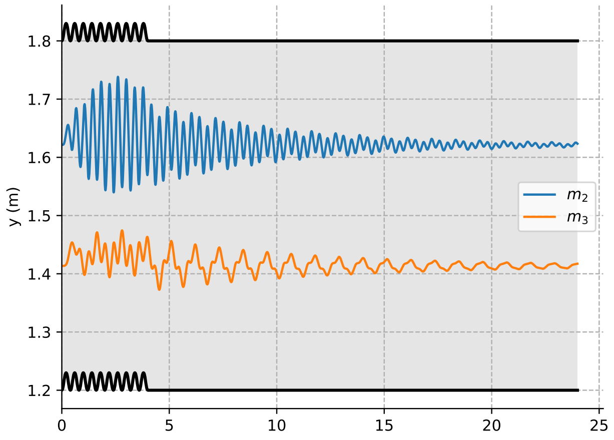
$$c_2 = \omega_1^2 \omega_2^2 \approx 9093\,\mathrm{Hz}^4$$

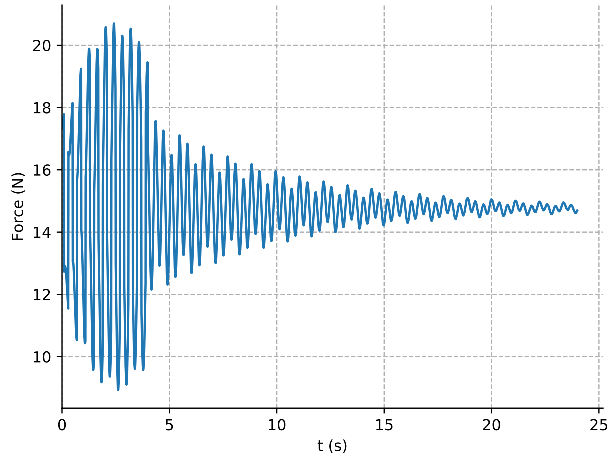
$$m_2 = \frac{k_1^2}{c_1 k_1 - c_2(m_2 + m_3)} \approx 0.238\,\mathrm{kg}$$

$$m_3 = 0.649\,\mathrm{kg} - 0.238\,\mathrm{kg} = 0.411\,\mathrm{kg}$$

$$k_2 = \frac{c_2 m_2 m_3}{k_1} \approx 22.4\,\mathrm{N/m}$$

Exact answers: $m_2 = 0.236\,\mathrm{kg}$, $m_3 = 0.413\,\mathrm{kg}$, $k_2 = 22.6\,\mathrm{N/m}$.





2.4.2 Method 2: Fast pulse

Similarly as in method 1 for finding k_1, we quickly accelerate the box with acceleration a_1 for time t and then decelerate with acceleration $-a_1$ for time t. If t is small, then y_2 does not change much while moving the box, so $\Delta F_1 = \Delta F \approx k_1 \Delta y_1$.

We also know that after the pulse when mass 2 starts to move, for a short time y_3 does not change much.

Therefore, $m_2 a_2 = \Delta F_1 - \Delta F_2 \approx \Delta F_1$ before mass 2 starts to significantly move, and $m_2 a_2 \approx k_1 \Delta y_1 - k_1 \Delta y_2 - k_2 \Delta y_2 = k_1 \Delta y_1 - (k_1 + k_2) \Delta y_2$ before mass 3 starts to significantly move.

Therefore, if t is small, then right after time t:

$$F - F_0 = \Delta F_1 \approx m_2 a_2 = m_2 \frac{d^2 y_2}{dt^2} = -\frac{m_2}{k_1} \frac{d^2 F_1}{dt^2} = -\frac{m_2}{k_1} \frac{d^2 F}{dt^2}.$$

This method is less accurate than the previous method, it does many approximations, ignores drag forces and resolution of F is small. It might be possible to make this method more accurate by taking the initial estimate for m_2 and then estimating Δy_2 while the box is accelerated to get a better estimate.

Theoretically it is also possible to find k_2, although it is even less accurate. When $\frac{d^2 F}{dt^2} = 0$, then $a_2 = 0$, which means that $k_1 \Delta y_1 \approx (k_1 + k_2) \Delta y_2$.

We will use $a_1 = 30\,\mathrm{m/s^2}$ for highest accuracy. A good trade-off between resolution of F and small t seems to be $t = 0.05$ s. Since we need to find the second derivative and need a lot of accuracy, we will average the values of 10 measurements. The results are shown in the table.

Time (s)	F (N)	$\frac{dF}{dt}$ (N/s)
0.10	12.572	
0.11	12.781	20.9
0.12	13.013	23.2
0.13	13.268	25.5
0.14	13.533	26.5
0.15	13.811	27.8
0.16	14.078	26.7
0.17	14.351	27.3
0.18	14.599	24.8

We estimate that $\frac{d^2F}{dt^2} \approx 230\,\mathrm{N/s^2}$ at $t = 0.11$ s. Therefore

$$m_2 \approx -\frac{(F - F_0)k_1}{\frac{d^2F}{dt^2}} = -\frac{(12.781 - 14.774)\,\mathrm{N} \cdot 39.7\,\mathrm{N/m}}{230\,\mathrm{N/s^2}}.$$

$$m_2 \approx 0.344\,\mathrm{kg}.$$

We also estimate that $\frac{d^2F}{dt^2} = 0$ at $t \approx 0.15$ s when $F = 13.811$ N. We know that $\Delta y_1 = a_1 t^2 = -30\,\mathrm{m/s^2} \cdot (0.05\,\mathrm{s})^2 = -0.075$ m.

At $F = 13.811$ N,

$$\Delta(y_1 - y_2) \approx \frac{(13.811 - 14.774)\,\mathrm{N}}{39.7\,\mathrm{N/m}} \approx -0.024\,\mathrm{m},$$

where we get $\Delta y_2 \approx 0.051$ m.

Since $k_1 \Delta y_1 \approx (k_1 + k_2)\Delta y_2$,

$$k_2 \approx k_1 \left(\frac{\Delta y_1}{\Delta y_2} - 1\right) \approx 18.4\,\mathrm{N/m}$$

	Finding m_2, m_3, k_2, Any method	
4a	Correct method	0.5
4b	Correct equations allowing to solve for the values m_2, m_3, k_2	0.5
	Correct equations allowing to solve only for m_2 and m_3	0.3
4c	Necessary measurements	1.0
	If only natural frequencies (periods) are found without a plan on how to use them:	
	T_1 in range 1.11 ± 0.02 s	0.3
	T_1 in range 1.11 ± 0.10 s	0.1
	T_2 in range 0.373 ± 0.005 s	0.3
	T_2 in range 0.373 ± 0.050 s	0.1
4d	k_2 in range 22.6 ± 0.5 N/m	1.0
	k_2 in range 22.6 ± 1.0 N/m	0.8
	k_2 in range 23 ± 3 N/m	0.6
	k_2 in range 23 ± 6 N/m	0.4
4e	m_2 in range 0.236 ± 0.010 kg or m_3 in range 0.413 ± 0.010 kg	0.9
	m_2 in range 0.236 ± 0.020 kg or m_3 in range 0.413 ± 0.020 kg	0.6
	m_2 in range 0.236 ± 0.050 kg or m_3 in range 0.413 ± 0.050 kg	0.3
4f	Correctly calculate m_3 using m_2 or vice versa given any points received in 4e	0.1
	Total:	**4.0**

Note: Equations of motion for mass 2 and 3 give 0 points. Getting a correct biquadratic equation for ω^2

gives 0.5 points from 4a, points are given for 4b only if k_2, m_2, or m_3 is correctly expressed from the biquadratic equation taking k_1, m_2+m_3, ω_1 and ω_2 as the only known parameters. Partial points can be given for getting an equation for ω^2 (0.4 p for slightly wrong result, 0.2 p for setting up the determinant). Finding T_1 and T_2 give 0.3/0.1 points even with a plan to use them to solve a system of equations but without an idea how. Having a biquadratic equation for ω^2 counts as "a plan" and in this case, finding T_1 and T_2 will each give 0.5/0.2 points with the same error tolerances. A correct answer without any justification or obtained with a physically nonsensible method gives 0 points.

2.4.3 Method 3: Estimating $\frac{k_2}{m_3}$ by estimating $y_2 - y_3$ at equilibrium and using $F_2 \approx k_2(y_2 - y_3)$

After finding $y_1 - y_2$ using method 3 to find k_1, we can similarly estimate $y_3 - (y_1 - a)$, where $a = 0.6\,\mathrm{m}$, by quickly accelerating the box upwards. This method assumes that the masses have negligible height (which is true).

Again, using binary search, we find that the time for collision is $t = 0.13\,\mathrm{s}$ at $a_1 = 25.6\,\mathrm{m/s^2}$. Thus

$$y_3 - (y_1 - a) \approx \frac{a_1}{2}t^2 \approx 0.216\,\mathrm{m}$$

$$y_2 - y_3 = a - (y_1 - y_2) - (y_3 - y_1 + a)$$

$$y_2 - y_3 \approx 0.6\,\mathrm{m} - 0.226\,\mathrm{m} - 0.216\,\mathrm{m} \approx 0.158\,\mathrm{m}$$

$$\frac{k_2}{m_3} \approx \frac{g}{y_2 - y_3} \approx 62.1\,\mathrm{N/(kg\,m)}$$

The actual values are $y_2 - y_3 = 0.208\,\mathrm{m}$ and $\frac{k_2}{m_3} = 54.7\,\mathrm{N/(kg\,m)}$.

Estimating k_2/m_3		
4.1a	Idea to use method	0.5
4.1b	Correctly estimate $y_3 - y_1 + a$	0.5
4.1c	Correct formula for k_2/m_3	0.2
4.1d	k_2/m_3 in range $55 \pm 10\,\mathrm{N/(kg\,m)}$	0.3
	Total:	**1.5**

Note: Estimating the distance $y_3 - y_1 + a$ without an idea how to use it gives 0 points.