T1: A Leak

Let p_1, V_1, T_1 denote the (time dependent) pressure, volume, and temperature in the upper chamber, and p_2, V_2, T_2 — those in the lower one. Note that $V_1 \equiv V$ does not change.

Consider a parcel of volume v below the diaphragm containing n moles of helium. It is convenient to imagine it bounded by two fictitious free thin massless pistons. During slow perturbations the parcel undergoes an adiabatic process. The pressure and the temperature for the parcel are actually the pressure and the temperature for entire lower chamber p_2 and T_2. The energy conservation for the parcel is

$$0 = p_2 \mathrm{d}v + \mathrm{d}\left(\frac{3}{2} n_v R T_2\right) = \frac{5}{2} p_2 \mathrm{d}v + \frac{3}{2} v \mathrm{d}p_2.$$

This gives

$$v^5 p_2^3 = \text{const}, \tag{1}$$

$$T_2^5 p_2^{-2} = \text{const}\,. \tag{2}$$

The leak begins when the pressure below the diaphragm exceeds that in the upper chamber by $\Delta p \equiv p_0 - p = mgH/V = px$, where

$$x = \frac{mgH}{pV}. \qquad \textbf{(a., b., c.)}$$

a. We may let $v = V_2$ before that.

$$V^5 p^3 = V_0^5 (p + \Delta p)^3 = V_0^5 p^3 (1 + x)^3$$

$$V_0 = V(1 + x)^{-3/5} \qquad \textbf{(a.)}$$

b. The energy conservation for the whole system:

$$0 = p_1 \mathrm{d}V_1 + \mathrm{d}\left(\frac{3}{2} n_1 R T_1\right) + p_2 \mathrm{d}V_2 + \mathrm{d}\left(\frac{3}{2} n_2 R T_2\right) =$$

$$\frac{5}{2}\left(p_1 \mathrm{d}V_1 + p_2 \mathrm{d}V_2\right) + \frac{3}{2}\left(V_1 \mathrm{d}p_1 + V_2 \mathrm{d}p_2\right) =$$

$$\frac{5}{2} p_2 \mathrm{d}(V + V_2) + \frac{3}{2}(V + V_2)\mathrm{d}p_2,$$

since the pressure above the diaphragm remains lower than that below by the same margin Δp during the later process and $\mathrm{d}V_2 = \mathrm{d}(V + V_2)$. Similarly to (1), we get

$$(V + V_2)^5 p_2^3 = \text{const}\,.$$

The pressure p_2' in the lower chamber when the piston touches the diaphragm is found from the equation

$$V^5 p_2'^3 = (V + V_0)^5 p_0^3 = V^5 \left(1 + \frac{1}{(1 + x)^{3/5}}\right)^5 p^3 (1 + x)^3,$$

$$p_2' = p\left(1 + (1 + x)^{3/5}\right)^{5/3}. \tag{3}$$

The pressure in the upper chamber at this moment is

$$p_1' = p_2' - \Delta p = p\left(\left(1 + (1 + x)^{3/5}\right)^{5/3} - x\right).$$

The temperature in the upper chamber is found from the equations of state $pV = nRT$ and $p'V = (2n)RT'$

$$T_1' = \frac{T}{2}\left(\left(1 + (1 + x)^{3/5}\right)^{5/3} - x\right). \qquad \textbf{(b.)}$$

The temperature and the pressure in the lower chamber are related by (2). Substituting (3) we get

$$T_2' = T\left(\frac{p_2'}{p}\right)^{2/5} = T\left(1+(1+x)^{3/5}\right)^{2/3}. \qquad \textbf{(c.)}$$

Preliminary grading scheme

a1	It's stated (or written as a formula) that the process is adiabatic	0.5
a2	Relation between V and p is found in adiabaic process	1.0
a3	Condition on when the diaphragm leaks	0.5
a4	Answer for V_0	1.0
b1	Energy conservation for the whole system in differential form. If conservation law is written only for one half but heat transfer is taken into account: 0.5 pts.	1.0
b2	Internal energy for a mono-atomic gas	0.5
b3	Usage of $V_1 = \text{const}$	0.2
b3	Usage of $p_2 - p_1 = \text{const}$	0.3
b4	Relation between V_2 and p_2	1.0
b5	Equation to find p_2' (or T_2') before the end	1.0
b6	Usage of $n' = 2n$	0.5
b7	Answer for T_1'	0.5
c1	Relation between T_1' and T_2'	1.0
c2	Answer for T_2'	1.0

Arithmetic or typo errors gives half of point (rounded up to 0.1) for the item and is not considered as a mistake afterwards.