T2: Thread on cylinder

Fig. 1 shows the cylinder and the loop from three different angles; P denotes the pulled point of the loop, while O is the top point of the thread.

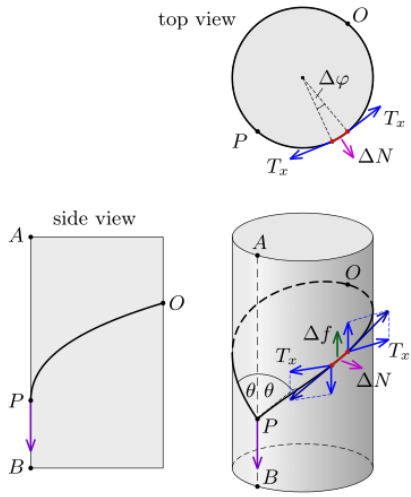


Fig. 1

Imagine that the side of the cylinder is cut along the generatrix AB passing through point P, and then the side (including the loop) is unfolded as shown in *Fig. 2*. In this figure points A and A', B and B', P and P' are equivalent, respectively. Let us introduce a Cartesian coordinate system on this unfolded plane so that point O is the origin, axis z is parallel with the axis of cylinder and directed downwards, axis x is perpendicular to z (i.e. horizontal).

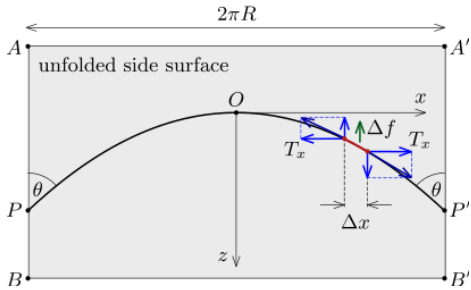


Fig. 2

Consider the forces acting on a small piece of the thread (with horizontal projection Δx) indicated with red line in both figures. These are the tensions at both ends of the small piece exerted by neighbouring parts of the thread, the normal force ΔN and the friction force Δf exerted by the cylinder. On the verge of slipping, the direction of Δf is parallel to the z-axis. Since the small piece of thread is in equilibrium, the x-component of the tension is the same everywhere:

$$T_x = \text{const.}$$

The normal force ΔN can be determined by looking at the top view of the loop in *Fig. 1*. The polar angle corresponding to small piece of thread is $\Delta\varphi = \Delta x/R$, so the force balance in the radial direction can be written as

$$2T_x \underbrace{\sin\frac{\Delta\varphi}{2}}_{\approx\Delta\varphi/2} - \Delta N = 0 \quad \longrightarrow \quad \Delta N = T_x \frac{\Delta x}{R}\,. \tag{4}$$

The frictional force on the verge of slipping is given by

$$\Delta f = \mu \Delta N \,. \tag{5}$$

Thus, the force balance on the small piece of thread in the z direction (see *Fig. 2*):

$$T_x \left.\frac{\mathrm{d}z}{\mathrm{d}x}\right|_{x+\Delta x} - T_x \left.\frac{\mathrm{d}z}{\mathrm{d}x}\right|_{x} - \Delta f = 0 \,, \tag{6}$$

where we expressed the z-component of the tension forces with T_x and the tangent $\mathrm{d}z/\mathrm{d}x$. Using the three equations above and taking the limit $\Delta x \to 0$, we get the differential equation

$$\frac{\mathrm{d}^2 z}{\mathrm{d}x^2} = \frac{\mu}{R} \,,$$

where we used the fact that $T_x \neq 0$. By direct integration and taking into account the boundary conditions $z(0) = 0$ and $z'(0) = 0$ we get

$$z(x) = \frac{\mu}{2R} x^2 \,,$$

so the shape of the thread on the unfolded side surface (*Fig. 2*) can be described by a parabola.

The thread needs to span over the entire cylinder, so its length can be calculated as

$$L_0 = \int_{-\pi R}^{\pi R} \sqrt{\mathrm{d}x^2 + \mathrm{d}z^2} = 2 \int_0^{\pi R} \sqrt{1 + \left(\frac{\mathrm{d}z}{\mathrm{d}x}\right)^2} \, \mathrm{d}x \,.$$

Substituting the $z(x)$ function we get:

$$L_0 = 2 \int_0^{\pi R} \sqrt{1 + \left(\frac{\mu x}{R}\right)^2} \, \mathrm{d}x = \left(\frac{R}{\mu}\right) 2 \int_0^{\pi \mu} \sqrt{1 + \xi^2} \, \mathrm{d}\xi \,,$$

where we introduced the notation $\xi = \mu x / R$. Using the integral given in the text of the problem:

$$L_0 = \pi R \sqrt{1 + (\pi\mu)^2} + \frac{R}{\mu} \operatorname{arcsinh}(\pi\mu) \,.$$

If the length of the thread is shorter than the length calculated here, then there is no solution satisfying the thread length constraint, i.e. the thread cannot slip.

Note. In the limit $\mu \to 0$ the loop slips even if $L = 2\pi R$, which should be reproduced by our final formula. Using the relation for the inverse hyperbolic function given in the problem text, then expanding the logarithm in Taylor series in linear order around 0 we get:

$$\operatorname{arcsinh} x \equiv \ln\left(x + \sqrt{1 + x^2}\right) \approx \ln(x + 1) \approx x \,,$$

so for small values of μ we get

$$L_0 \approx \pi R + \frac{R}{\mu}(\pi\mu) = 2\pi R \,.$$

Grading scheme: T2	
2-i. A figure or figures reflecting the correct geometry such as: the loop is a non-planar curve in all figures (0.5 p), existence of exactly one cusp (0.5 p, if the „back" is hidden, 0 p).	0.5 p + 0.5 p
2-ii. Realizing that on the verge of slipping the frictional force is parallel with the axis of the cylinder for every small piece of the thread and $\Delta f = \mu \Delta N$. (If any of the two is missing, then 0.2 p)	0.5 p
2-iii. Correct equation for the force balance in z direction involving frictional force or the load (which is pulling down the loop) and the z components of the tension.	0.5 p
2-iv. T_x = const. + correct explanation based on the balance in x direction. If the physics is incompatible with the geometrical assumptions (e.g. planar curve for the loop and the existence of a frictional force acting on small pieces), no points are given	0.5 p + 1.5 p
2-v. Expressing the normal force ΔN acting on a small segment with T_x and $\mathrm{d}x$ (or $\mathrm{d}\phi$). If the relation $x = R\varphi$ or $\mathrm{d}x = R\mathrm{d}\varphi$ is not used here or anywhere else, 1.5 p is given. If the expression is wrong but ΔN is proportional to the curvature $(1/R)$, 1.0 p is given.	2.0 p
2-vi. Deriving the correct differential equation for $z(x)$ (if the diff. equation is wrong due to any reason, 0 p)	1.5 p
2-vii. Solving the diff. equation for $z(x)$ correctly (including boundary conditions). If only one integral is evaluated correctly, 0.3 p are given. For stating only the boundary conditions (both $z(0)$ and $z'(0)$) 0.2 p.	1.0 p
2-viii. Writing down the length of the thread in terms of an integral of $z(x)$, i.e. writing down the length constraint	0.5 p
2-ix. Evaluating the integral correctly (factor mistake in calculation 0.5 p, wrong units 0.2 p)	1.0 p
Total T2:	**10.0 p**

General guidelines for marking:

- Granularity for marks is 0.1 p.
- A simple numerical error resulting from a typo is punished by 0.2 p unless the grading scheme explicitly says otherwise.
- Errors which cause dimensionally wrong results are punished by at least 50% of the marks unless the grading scheme explicitly says otherwise.
- Propagating errors are not punished repeatedly unless they either lead to considerable simplifications or wrong results whose validity can easily be checked later.