T3: Glass ball

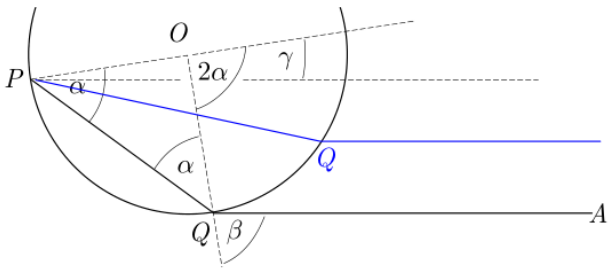
To begin with, let us notice that if a ray coming from a point P on the stripe is refracted at point Q at the surface of the ball towards a very distant point A (which denotes the aperture of the camera lens), the ray will remain in the plane PQO where O is the centre of the sphere. This means that those rays which arrive from P to A must lay in the plane POA, and the rays can be conveniently depicted in the POA-plane.

The angle $\gamma = \gamma(\alpha)$ between vectors PO and QA is given by

$$\gamma \equiv 2\alpha - \beta \tag{7}$$

$$n \sin\alpha = \sin\beta \tag{8}$$

and is a non-monotonous function of α which achieves its maximum γ_0 by a certain α_0. This means that for a fixed P and A, if $\angle OPA < \gamma_0$, there are two such angles α and, hence, two such positions Q_1 and Q_2 for the point Q that the ray PQA will reach the point A (here we have assumed that the point A is at a very big distance). So, when viewed from the point A, the image of the point P is split into two points Q_1 and Q_2. On the other hand, if $\angle OPA > \gamma_0$, rays from P cannot reach A, and such points on the stripe cannot be seen in the photo. If $\angle OPA = \gamma_0$, the two images Q_1 and Q_2 merge into a single point bridging the two images of a piece of stripe into a closed loop.



Now it becomes clear that the place where images Q_1 and Q_2 merge into a single image Q_0 is the key to finding the coefficient of refraction. Indeed, the point Q_0 can be found in the photo as the point where the blue ellipse (the images of a segment of the stripe in red light as the blue shadow is where the red light is missing) is touching a radius of the ball, see the figure below (we need to find tangent point with the radius because we need to consider plane QOA which projects into a line through the ball's centre). There is no difference between taking the tangent to the outer edge of the elliptical stripe and taking the tangent to the inner edge of it (point Q_0' in the photo below). Then, $\sin\beta_0 = n \sin\alpha_0$ can be determined from the photo as the ratio of the lengths h and R, $h/R \approx 0.765$, where h denotes the distance of the ball's centre O from the line Q_0A, measurable in the photo as the length OQ_0 (when looking from the distant point A, we can see only the perpendicular-to-OA component of the segment OQ_0, and R is the ball's radius.

Since we look for an extremum of γ, upon taking differentials from Eqns. (7,8), we obtain

$$2\,\mathrm{d}\alpha = \mathrm{d}\beta$$

$$n\cos\alpha\,\mathrm{d}\alpha = \cos\beta\,\mathrm{d}\beta$$

from where

$$n\cos\alpha = 2\cos\beta. \tag{9}$$

$$n = \sqrt{\sin^2\beta + 4\cos^2\beta} = \sqrt{4 - 3\sin^2\beta} \approx 1.498 \approx 1.50.$$

In order to find Δn, we could find in a similar way n_V ($|OQ_0''| \approx 0.755R$, $n \approx 1.513 \approx 1.51$), but the result would have a huge relative uncertainty as $|OQ_0|$ and $|OQ_0''|$ have very similar lengths. A much more precise result will be obtained if we base our calculations on the segment length $|ST|$, see the figure below. Q_0'' is where the blue rays have an extremum for γ while Q_0' and Q_0'' are two red rays originating from the same point on the stripe. From the photo we can measure $|ST| \approx 0.20R$, hence we can use the small parameter $SQ_0''/R \approx 0.10$.

Rays from S, T, and Q_0'' arrive to the lens aperture, so all these rays (which originate from the same point P on the stripe) have the same value of γ. So, we have a set of equations

$$\gamma \equiv 2\alpha_R - \beta_R = 2\alpha_V - \beta_V \tag{10}$$

$$n_V \sin\alpha_V = \sin\beta_V \tag{11}$$

$$(n_V - \Delta n)\sin\alpha_R = \sin\beta_R \tag{12}$$

$$n_V \cos\alpha_V = 2\cos\beta_V, \tag{13}$$

and we would like to get an expression relating Δn to $\sin\beta_{V1} - \sin\beta_{V2}$, where the indices 1 and 2 relate to the two different solutions. Expressing $\alpha_R = \alpha_V + \delta$ we obtain from Eq. (10) that $\beta_R = \beta_V + 2\delta$. Now, if we expand Eq. (12) into Taylor series, neglect the smallest term with $\Delta n\delta$, and keep in mind Eq. (11), we obtain

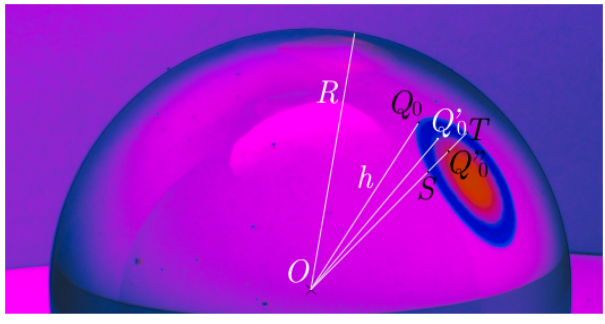
$$n_V\delta\cos\alpha_V - n_V\delta^2\frac{\sin\alpha_V}{2} - \Delta n\sin\alpha_V = 2\delta\cos\beta_V - 2\delta^2\sin\beta_V.$$

Here the two linear-in-δ terms cancel out due to Eq. (13) so that with $\sin\beta_V = n_V\sin\alpha_V$ we obtain

$$\Delta n = 3n_V\delta^2/2. \tag{14}$$

On the other hand, $|ST|/R = \sin\beta_{R1} - \sin\beta_{R2} \approx \sin(\beta_V + 2\delta) - \sin(\beta_V - 2\delta) = 4\delta\cos\beta_V$, hence

$$\Delta n = \frac{3n_V}{2}\left(\frac{|ST|}{4R\cos\beta_V}\right)^2 = \frac{3n_V}{32}\frac{|ST|^2}{R^2 - h^2} \approx 0.0132 \approx 0.013. \tag{15}$$



Grading scheme

A-i

- Proving that we can see a loop because for a range of points on the thread, a single point of thread creates two images: 0.7 pts
- The endpoints of that thread create one image forming thereby a closed loop: 0.3 pts

Else if only the fact that rays coming to A must be in the PQO plane is stated: 0.3 pts

A-ii Drawing a ray diagram where we can see that two rays originating from a single point arrive both to the lens while crossing the ball's surface at different points and following the Snell's law: 1.0 pt

Else if such idea is demonstrated by words or by a rough sketch (which does not obey Snell's law): 0.5 pts

B-i Relating angle β to the measurable distance h in the photo (any point on the blue or red ellipse will earn the mark): 1pt

B-ii Measuring ratio of h to R: 0.5 pts

B-iii The idea of using point Q_0: 1 pt (using Q_0' will earn only 0.5 pts as this corresponds to the violet light)

B-iv Obtaining a correct expression of γ as a function of α (or equivalent calculations in different parametrisation): 1 pt

B-v Finding extremum and hence, the final expression of n as a function of β or h: 1pt. If an incorrect expression is obtained from reasonable physics because of mistakes in algebraic manipulation: 0.5 pts

B-vi If the numerical answer is correct within ± 0.03 and is found using reasonable physics: 0.5 pts. Else if the numerical answer is correct within ± 0.1 and is found using reasonable physics: 0.3 pts. No point if the answer is guessed or is found using completely wrong or irrelevant physics.

C-i The idea of using the width or length of the blue ellipse: 1 pt (red ellipse cannot be used as we don't know if the center of the red ellipse corresponds to the edge of the thread, or to a point at its middle)

C-ii Obtaining Eq (14) or something equivalent based on considering neighbouring red and violet rays: 1 pt. If an incorrect expression is obtained from reasonable physics because of mistakes in algebraic manipulation: 0.5 pts

C-iii Expressing Δn correctly in terms of measurable quantities: 0.5 pts

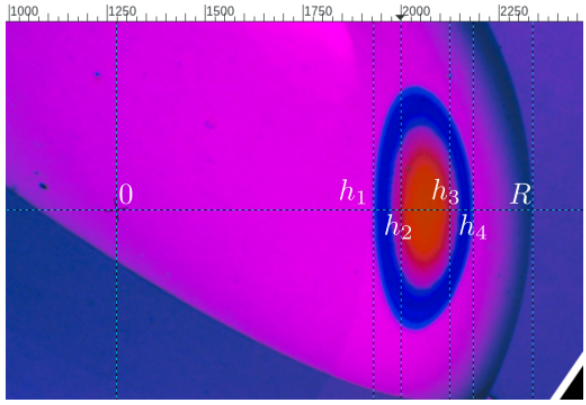
C-iv Finding numerical answer correct within ± 0.003: 0.5 pts. (0 pts if this answer is obtained without considering the width or height of the blue ellipse.)

Solution 2

One may alternatively prefer to consider a plane, perpendicular to the thread and containing the ball centre O and the camera A. This is the plane of symmetry for the system and corresponds to the maximum width of the ovals. The same arguments as in the first Solution lead to

$$\gamma = \beta - 2\arcsin\frac{\sin\beta}{n} = \arcsin\frac{h}{R} - 2\arcsin\frac{h}{nR}.$$

For $n < 2$ this function is not monotonous and may give the same result for two different values of h (that is why the mapping of an arc is a closed oval).



To get the refraction index for red we solve the equation

$$\arcsin\frac{h_1}{R} - 2\arcsin\frac{h_1}{n_R R} = \arcsin\frac{h_4}{R} - 2\arcsin\frac{h_4}{n_R R},$$

which gives $n_R \approx 1.51$.

One could repeat the previous argument to find n_V. One could also compare the images of the same thread elements in different colours. This gives 4 equations

$$\arcsin\frac{h_{1,4}}{R} - 2\arcsin\frac{h_{1,4}}{n_R R} = \arcsin\frac{h_{2,3}}{R} - 2\arcsin\frac{h_{2,3}}{n_V R},$$

where $h_{1,4}$ means we can replace it with the value of h_1 or h_4. Similar rule applies for $h_{2,3}$. From each equation, we can solve for one value of n_V and then take their average. This results in $n_V - n_R \approx 0.0145$.

Grading scheme

A-i

- Proving that we can see a loop because for a range of points on the thread, a single point of thread creates two images: 0.7 pts
- The endpoints of that thread create one image forming thereby a closed loop: 0.3 pts

Else if only the fact that rays coming to A must be in the PQO plane is stated: 0.3 pts

A-ii Drawing a ray diagram where we can see that two rays originating from a single point arrive both to the lens while crossing the ball's surface at different points and following the Snell's law: 1.0 pt

Else if such idea is demonstrated by words or by a rough sketch (which does not obey Snell's law): 0.5 pts

B-i Relating angle β to the measurable distance h in the photo (any point on the blue or red ellipse will earn the mark): 1pt

B-ii Measuring ratio of h to R: 0.5 pts

B-iii The idea of using points on the blue ellipse lying on the same radius of the sphere: 1 pt

B-iv Obtaining the correct equation to solve for n_R numerically: 2 pts. If an incorrect equation is obtained from reasonable physics because of mistakes in algebraic manipulation: 1 pt

B-v If the numerical answer is correct within ± 0.03 and is found using reasonable physics: 0.5 pts. Else if the numerical answer is correct within ± 0.1 and is found

using reasonable physics: 0.3 pts. No point if the answer is guessed or is found using completely wrong or irrelevant physics.

C-i The idea of using points on the red (or both) ellipse(s) lying on the same radius of the sphere: 1 pt

C-ii Obtaining at least one correct equation to solve for n_V numerically: 1.5 pts. If an incorrect equation is obtained from reasonable physics because of mistakes in algebraic manipulation: 1 pt

C-iii Finding numerical answer correct within ± 0.003: 0.5 pts.