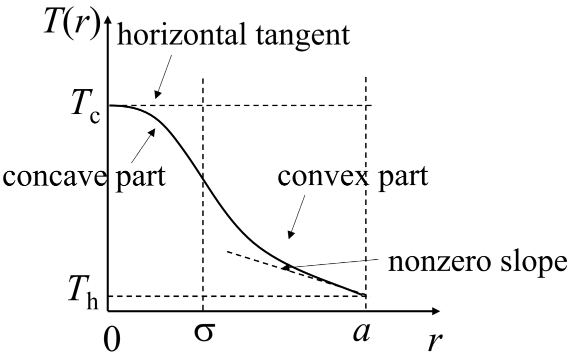
T1: Thermal lens - Solution

(a) Drawing a $T(r)$ graph

The graph should present or clearly infer for the four elements shown in the figure.



Element of the graph	Pts.
Horizontal tangent at $r = 0$	0.5
Graph is concave and decreasing in $0 \le r \le \sigma$	0.5
Graph is convex and decreasing in $\sigma \le r \le a$	0.5
Nonzero negative slope at $r = a$	0.5
Totally on (a)	**2.0**

(b) Finding T_c

Approach with a direct solution of the heat-transport equation.

Consider a cylindrical cut of the disk with radius r. Let $P_{\text{abs}}(r)$ be the portion of laser power absorbed within the cylinder. The absorbed power is being transferred as heat towards the outer holder through the circumvent surface $2\pi br$ of the cylinder. The heat-transport equation reads:

$$-k(2\pi rb)\frac{\mathrm{d}T(r)}{\mathrm{d}r} = P_{\text{abs}}(r) \qquad (1)$$

Depending on r, the absorbed power is given by two different expressions. Within the illuminated area $0 \le r \le \sigma$, the incident light intensity $I = P_{\text{L}}/(\pi\sigma^2)$ is constant, so the absorbed power is:

$$P_{\text{abs}}(r) = I\pi r^2 = AP_{\text{L}}\frac{r^2}{\sigma^2} \qquad (2)$$

and the solution for $T(r)$ is quadratic in r:

$$T(r) = T_{\text{c}} - \frac{AP_{\text{L}}r^2}{4\pi kb\sigma^2} \qquad (3)$$

where $T_{\text{c}} = T(0)$ is the temperature at the center. It is clear from that expression that the parameter m is:

$$m = -\frac{AP_{\text{L}}}{4\pi kb\sigma^2} = -1.1\cdot 10^7\,\text{Km}^{-2} \qquad (4)$$

Outside the illuminated area, i.e. for $\sigma \le r \le a$, $P_{\text{abs}}(r) = AP_{\text{L}}$, which does not depend on r. The solution for $T(r)$ is logarithmic in r:

$$T(r) = T_{\text{h}} + \frac{AP_{\text{L}}}{2\pi kb}\ln\left(\frac{a}{r}\right) \qquad (5)$$

where $T_{\text{h}} = T(a)$ is the temperature of the holder, which is equal to the temperature along the outer rim of the disk. After matching the two solutions at $r = \sigma$ we obtain:

$$T_{\mathrm{c}} = T_{\mathrm{h}} + \frac{AP_{\mathrm{L}}}{4\pi kb}\left[1 + 2\ln\left(\frac{a}{\sigma}\right)\right] = 41°\mathrm{C} \tag{6}$$

Task	**Pts.**
Writes down the heat-transport equation in radial coordinates	0.7
Derives an expression for P_{abs} at $0 \le r \le \sigma$	0.5
Finds a quadratic solution for $T(r)$ in the region $0 \le r \le \sigma$ (Subtract 0.2 pts. if he boundary condition $T(0) = T_{\mathrm{c}}$ has not been accounted for)	0.5
Identifies the expression for m (Subtract 0.1 pts. if the minus sign is missing)	0.2
Calculates m numerically (Subtract 0.1 pts. if the minus sign is missing)	0.2
Derives an expression for P_{abs} at $\sigma \le r \le a$	0.3
Finds a logarithmic solution for $T(r)$ in the region $\sigma \le r < a$ (Subtract 0.2 pts. if he boundary condition $T(a) = T_{\mathrm{h}}$ has not been accounted for)	0.5
Sets up an equation by matching the two solutions at $r = a$	0.6
Writes down the final expression for T_{c}	0.2
Calculates T_{c} numerically	0.3
Totally on (b)	**4.0**

Alternative approach – direct piece-wise integration of the heat-transport equation.

After realizing that $P_{\mathrm{abs}}(r)$ is given by a piece-wise function:

$$P_{\mathrm{abs}} = \begin{cases} AP_{\mathrm{L}} r^2/\sigma^2 & \text{if } 0 \le r \le \sigma \\ AP_{\mathrm{L}} = \mathrm{const} & \text{if } \sigma \le r \le a \end{cases} \tag{7}$$

the student may substitute the given solution $T(r) = T_{\mathrm{c}} + mr^2$ into heat-transport equation (1) for $0 \le r \le \sigma$. This gives directly the expression (4) for the parameter m. The parameter T_{c} could be easily identified with the temperature $T(0)$ at the center of the disk. On the other hand, $T_{\mathrm{h}} = T(a)$ due to the thermal contact between the rim of the disk and the holder. It follows from the heat-transport equation that:

$$-\frac{\mathrm{d}T(r)}{\mathrm{d}r} = \frac{P_{\mathrm{abs}}(r)}{2\pi kbr} \tag{8}$$

The piece-wise integration of the two sides of the equation in the interval $0 \le r \le a$ gives:

$$\begin{aligned} T(0) - T(a) = T_{\mathrm{c}} - T_{\mathrm{h}} &= \int_0^a \frac{P_{\mathrm{abs}}(r)}{2\pi kbr}\mathrm{d}r \\ &= \int_0^\sigma \frac{P_{\mathrm{abs}}(r)}{2\pi kbr}\mathrm{d}r + \int_\sigma^a \frac{P_{\mathrm{abs}}(r)}{2\pi kbr}\mathrm{d}r \\ &= \frac{AP_{\mathrm{L}}}{4\pi kb}\left[1 + 2\ln\left(\frac{a}{\sigma}\right)\right] \end{aligned} \tag{9}$$

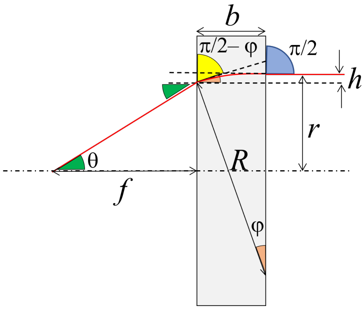
which is equivalent to the expression (6) for T_{c}.

Task	Pts.
Writes down the heat-transport equation in radial coordinates	0.7
Derives an expression for P_{abs} at $0 \le r \le \sigma$	0.5
Derives an expression for P_{abs} at $\sigma \le r \le a$	0.3
Substitutes the given form of $T(r)$ into heat-transport equation for the region $0 \le r \le \sigma$	0.3
Obtains the expression for m (Subtract 0.1 pts. if the minus sign is missing)	0.2
Calculates m numerically (Subtract 0.1 pts. if the minus sign is missing)	0.2
States that $T_c = T(0)$	0.2
States that $T_h = T(a)$	0.2
Expresses $T_c - T_h$ through integral of $\mathrm{d}T/\mathrm{d}r$ in the $0 \le r \le a$ interval	0.3
Calculates the integral in the $0 \le r \le \sigma$ interval	0.3
Calculates the integral in the $\sigma \le r \le a$ interval	0.3
Writes down the final expression for T_c	0.2
Calculates T_c numerically	0.3
Totally on (b)	**4.0**

(c) Finding the focal length

Approach based on the Fermat's principle

We consider only the illuminated area of the disk ($0 \le r \le \sigma$). Due to the nonuniform temperature distribution, the index of refraction is also r-dependent, which leads to a bending of the light rays incident at nonzero radii r, as shown schematically in the figure. As a result, the light rays exiting the disk, converge toward the optical axis, and, eventually, cross it in a certain point at a distance f from the disk.



Since the ray bending is relatively small, one may assume that: (i) all the rays travel approximately the same distance b inside the disk; (ii) any given ray enters and exits the disk at approximately the same height r; (iii) $f \gg r$. Thus, the optical pathway $s(r)$ of a ray, incident at a height r above the optical axis, is:

$$s(r) \approx n(r)b + \sqrt{f^2 + r^2} \approx n(r)b + f + \frac{r^2}{2f} \tag{10}$$

According to Fermat's principle, in order that all the rays focus at the same point, it is necessary that $s(r)$ is constant within $0 \le r < \sigma$. In particular, $s(r) \equiv s(0)$ for any r, which leads to the condition:

$$b(n(0) - n(r)) \equiv \frac{r^2}{2f} \tag{11}$$

Since:

$$n(0) - n(r) = \gamma(T(0) - T(r)) = -\gamma m r^2 = \gamma |m| r^2 \tag{12}$$

condition (11) is satisfied if

$$\gamma b|m|r^2 \equiv \frac{r^2}{2f} \tag{13}$$

The beam hence focuses at

$$f = \frac{1}{2\gamma b|m|} \tag{14}$$

Taking into account the expression for m derived in part (b), we represent the answer in terms of the known parameters and calculate its numerical value:

$$f = \frac{2\pi k\sigma^2}{\gamma AP} \approx 0.94\,\mathrm{m} \tag{15}$$

Alternatively, the student may state that the optical pathway $s(r)$ does not depend on r and use the condition $\mathrm{d}s/\mathrm{d}r \equiv 0$, which gives:

$$\frac{\mathrm{d}n(r)}{\mathrm{d}r}b + \frac{r}{f} \equiv 0 \tag{16}$$

Since:

$$\frac{\mathrm{d}n(r)}{\mathrm{d}r} = \frac{\mathrm{d}n}{\mathrm{d}T}\frac{\mathrm{d}T}{\mathrm{d}r} = \gamma 2mr \tag{17}$$

we obtain:

$$\frac{r}{f} + 2b\gamma mr \equiv 0 \tag{18}$$

The beam thus focuses at:

$$f = \frac{1}{2\gamma b|m|} = \frac{2\pi k\sigma^2}{\gamma AP} \approx 0.94\,\mathrm{m} \tag{19}$$

Task	**Pts.**
Formulates assumptions (i) and (ii) or equivalent statements	0.2
Derives a general expression for the optical pathway s as a function of r	0.8
Uses that $f \gg r$ and derives approximate quadratic expression for $s(r)$	0.5
States that focusing takes place when $s(r)$ is the same for all rays converging in the focus	0.5
Writes explicitly equation in the form $s(r) \equiv s(0)$ OR $\mathrm{d}s/\mathrm{d}r \equiv 0$	0.5
Uses the $T(r)$ dependence to derive the $n(r)$ dependence OR to find $\mathrm{d}n/\mathrm{d}r$	0.5
Derives an expression for the focal length f in terms of m or of the parameters given in the problem statement	0.8
Calculates f numerically	0.2
Totally on (c)	**4.0**

Approach based on a direct ray/wavefront tracing

As a next approximation, the light ray inside the disk can be modeled as a circular arc of a radius R ($R \gg b$, see the figure). As a result, the ray exits the disk at a smaller height $r - h$ above the optical axis ($h \ll r$). The angle of bending φ of the ray inside the material is related to h by:

$$\cos\varphi = 1 - \frac{h}{R} \tag{20}$$

From the Snell's law it follows that:

$$n(r)\sin(\pi/2) = n(r-h)\sin(\pi/2-\varphi) = n(r-h)\cos\varphi \quad (21)$$

Up to terms, linear in h, one may write that:

$$\cos\varphi = \frac{n(r)}{n(r-h)} \approx 1 + \frac{n'(r)}{n(r)}h \quad (22)$$

Thus, the radius of the ray inside the material, is:

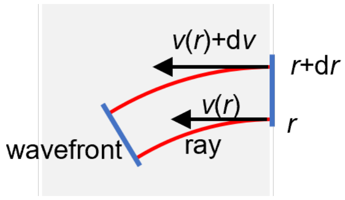
$$R = -\frac{n(r)}{n'(r)} \quad (23)$$

and the bending angle is approximately:

$$\varphi = \frac{b}{R} = -\frac{n'(r)b}{n(r)} \quad (24)$$

Alternatively, the students may trace a small part of the wavefront, associated with two rays, incident at close distances r and $r + \mathrm{d}r$ from the optical axis. By noticing that the wavefront is perpendicular to the rays, the angle φ of deviation of the rays is equal to the angle of rotation of the wavefront. Let $v(r) = c/n(r)$ be the speed of light in the material at a distance r from the axis. The time-rate $\dot{\varphi}$, i.e. the angular speed of the wavefront is:

$$\dot{\varphi} = \mathrm{d}v(r)/\mathrm{d}r = -cn'(r)/n(r)^2 \quad (25)$$



The rays reach the opposite surface of the disk in approximately the same time $t = bn(r)/c$, so the total deflection angle of the wavefront, and of the rays, thereof, is $\varphi = \dot{\varphi}t = -n'(r)b/n(r)$

Upon exiting the disk, the ray undergoes additional refraction, and inclines at a new angle θ relative to the optical axis. The Snell's law in the small-angle approximation ($\sin\theta \approx \theta$, $\sin\varphi \approx \varphi$) states that:

$$\theta = n(r-h)\varphi \approx n(r)\varphi = -n'(r)b \quad (26)$$

Since $n'(r) = \gamma T'(r) = 2mr$, one obtains the following expression for the angle of inclination:

$$\theta = 2\gamma|m|r \quad (27)$$

It is clear from the figure that:

$$f = \frac{r-h}{\tan\theta} \approx \frac{r}{\theta} = \frac{1}{2\gamma|m|b} \quad (28)$$

which reproduces the result obtained by the Fermat's principle.

Task	Pts.
States or shows on a graph that the light ray inside the material can be approximated by an arc OR illustrates on a graph that the wavefront is perpendicular to the light rays	0.2
Derives the relation $\cos\varphi = 1 - h/R$ or equivalent OR expresses the time of travel of the ray across the disk	0.5
Applies the Snell's law to the ray path inside the material OR derives a formula for the time-rate $\dot{\varphi}$ by considering the wavefront passing through two closely separated rays	0.8
Finds expression for the bending angle φ inside the material	0.5
Applies the Snell's law for the refraction of rays exiting the disk	0.5
Uses the $T(r)$ dependence to find $n'(r)$	0.5
Derives an expression for the focal length f in terms of m or of the parameters given in the problem statement	0.8
Calculates f numerically	0.2
Totally on (c)	**4.0**

Alternatively:
Consider a ray incident at a specific height r_0. Let $x \in [0, b]$ be the horizontal coordinate of the ray inside the material. Our goal is to find approximately the profile $r(x)$ of the light ray inside the material. From Snell's law:

$$n(r)\cos\varphi = n(r_0)$$

Since:

$$\tan\varphi = -\frac{\mathrm{d}r(x)}{\mathrm{d}x} = -\frac{\sqrt{(1-\cos^2\varphi)}}{\cos\varphi}$$

we obtain:

$$-\frac{\mathrm{d}r(x)}{\mathrm{d}x} = \frac{\sqrt{(n(r)-n(r_0))(n(r)+n(r_0))}}{n(r)}$$

Since $n(r) \approx n(r_0) + n'(r_0)(r - r_0) = n(r_0) + 2\gamma|m|r_0(r_0 - r)$, the differential equation approximates to:

$$-\frac{\mathrm{d}r(x)}{\mathrm{d}x} = \sqrt{\frac{4\gamma|m|r_0}{n(r_0)}}\sqrt{r0 - r}$$

which is solvable by separation of variables and gives an approximate parabolic path:

$$r(x) = r_0 - \frac{\gamma|m|r_0}{n(r_0)}x^2$$

Finally, for the angle, just before exiting the disk, we obtain:

$$\varphi \approx \tan\varphi = -r'(x=b) = \frac{2\gamma|m|r_0 b}{n(r_0)}$$

Task	**Pts.**
Applies the Snell's law to express the angle φ dependence on r for one ray	0.8
Expresses $\tan\varphi = dr/dx$	0.2
Applies approximation to derive a separable differential equation for $r(x)$	0.8
Integrates to obtain parabolic path	0.5
Finds expression for the angle φ, before exiting the disk	0.2
Applies the Snell's law for the refraction of rays exiting the disk	0.5
Derives an expression for the focal length f in terms of m or of the parameters given in the problem statement	0.8
Calculates f numerically	0.2
Totally on (c)	**4.0**