E1- Solution

The term "neuron" has been chosen in analogy to the cells of the nervous system, which transmit electrochemical signals across the nerves (variable y) depending on the integral stimulus (z) on their input extremities (variables x_i). The theory behind the artificial neurons and artificial neural networks has been developed in a close analogy to basic physical concepts. In recognition of this fact, the Nobel Prize in physics in 2024 was awarded to John J. Hopfield and Geoffrey E. Hinton "for foundational discoveries and inventions that enable machine learning with artificial neural networks". The artificial neurons could also be real physical systems, which transform mechanical, electrical, or optical signals.

Task 1

The multimeter is connected in an ohmmeter mode to the three possible pairs of output terminals of A-potentiometer. By turning the knob of the potentiometer we measure the maximum and minimum resistance for each pair:

Terminals:	$R_{\text{min}}(\Omega)$	$R_{\text{max}}(\Omega)$
A1-A2	1000	1000
A1-A3	222	1222
A2-A3	222	1222

If the load resistor was connected to either A1 or A2, then the R_{min} for A1-A3 and A2-A3 pairs would close to zero, in contrast to measurements. Therefore the load resistor is connected between A3 and the potentiometer slider. In this case, for either of A1-A3 and A2-A3 pairs we have: $R_{\text{min}} = R_L$ and $R_{\text{max}} = R_P + R_L$ and

$$R_L = 222\,\Omega \quad R_P = 1000\,\Omega$$

	Task 1	Pts
A	States or shows in a drawing that the resistance between the three pairs of terminals has been measured.	0.2
	Results for the resistances:	
B	$R_P \approx 1000\,\Omega \pm 120\,\Omega$	0.1
C	$R_L \approx 220\,\Omega \pm 6\,\Omega$	0.1
D	Concludes that the resistor is connected to terminal 3.	0.1
	Total on Task 1	0.5

Task 2

The necessary connections are shown in Fig. 1.

	Task 2	Pts
A	Potentiometers are connected to X1 and X2	0.2
	Correct connections to ground and supply	0.3
	-0.1 per wrong connection -0.2 fixed penalty for the specific case that X1 / X2 are connected to A1/A2 and V+ and GND on the remaining terminals of the potentiometer. This setup gives only limited variation of the input voltage ($0\,\mathrm{V}$ - $2.67\,\mathrm{V}$) -0.5 if only 2 terminals the potentiometers are connected.	
	Total on Task 2	0.5

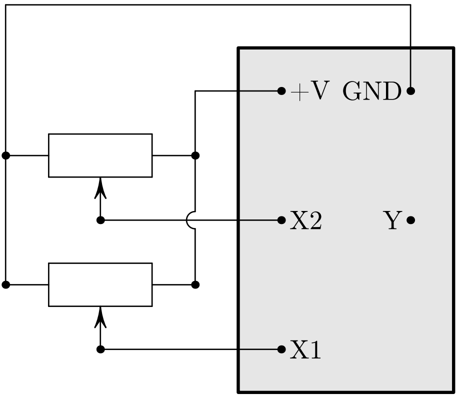

Figure 1: Correct Setup Task 2

Task 3

By turning the potentiometers' knobs one can establish that the input voltages change independently between $0\,\mathrm{V}$ and $\approx 3.25\,\mathrm{V}$ (Note that due to the protection circuit inside the box, this voltage decreases slightly with connected circuitry). Therefore, any combination of input voltages could be mapped to a point inside the shaded square area in the $x_1 - x_2$ plane, as shown in Fig 2. Points corresponding to a

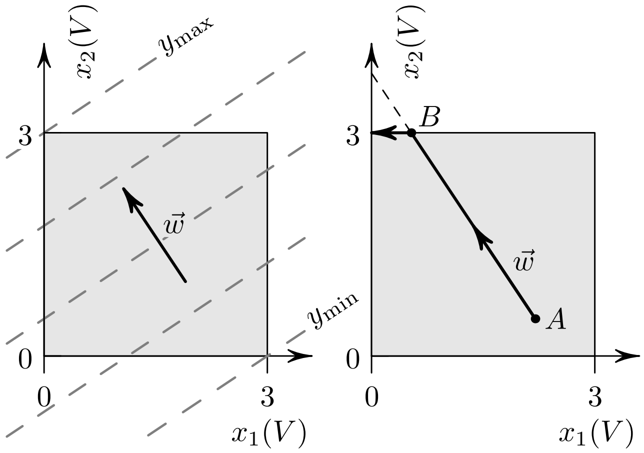

Figure 2: Task 3

$x_1(V)$	$x_2(V)$	$y(V)$
0.00	0.00	2.30
3.15	0.00	1.15
3.15	3.15	2.56
0.00	3.15	2.73

Table 1: Output voltage at the vertices

constant output voltage y satisfy the equation:

$$w_1x_1 + w_2x_2 + b = const,$$

which defines a family of parallel straight lines, perpendicular to the vector $\vec{w} = (w_1, w_2)$ (see the dashed lines in Fig. 2). It is clear that for any set of weights, the maximum voltage y_{max} and the minimum voltage y_{min} are always met at the vertices of the rectangle. Therefore, only three corners need to be measured in order to determine the maximum voltage y_{max}. First, two corners are measured, which are next to each other. Therefore, the maximum along the connecting edge can be found. Using the corner with the higher voltage, the other edge from that corner is followed to a third corner. Consequently, the highest voltage can be determined, as the fourth corner needs to be lower than the third measured corner. Table 1 summarizes the measurements of the output voltage at the four vertices. The maximum of the output voltage is:

$$y_{\text{max}} = 2.73\,\text{V}$$

corresponding to the input voltages:

$$x_1 = 0.00\,\text{V}; \quad x_2 = 3.15\,\text{V}.$$

	Task 3	Pts
A	Strategy that three measurements are sufficient to obtain $y_{\max}$	0.7
	Penalty for four measurements	-0.3
	No points for more than four measurements	
	Note that turning the knob for x_i and watching the change of y does count as at least 2 measurements per x_i. For solutions where the knob is turned a little for x_1, then the other for x_2, the conclusion about the signs of the w_i is made and only then the input gets adjusted to reach y_{max}, this counts as at least 5 measurements, which is awarded no points here.	
B	Lower bound of measurements $< 0.02\,\mathrm{V}$	0.1
C	Upper bound of measurements $> 3\,\mathrm{V}$	0.2
	Note that to obtain these 0.1 + 0.2 points, the student must write its input values for x_1 and x_2 down explicitly, or, at least write that they are directly connected to GND or +V if that is the case.	
D	Per measurement 0.1 up to 0.3	0.3
E	Value for $y_{\max} = 2.73\,\mathrm{V} \pm 0.04\,\mathrm{V}$	0.2
	Penalty for more than three significant digits	-0.2
	Penalty for only one significant digit	-0.2
	Total on Task 3	1.5

Task 4

The three parameters could be determined in two series of measurements of the output voltage by setting each of the input voltages constant and changing the other input voltage. The dependence of the output voltage on x_1 and x_2 can be linearized by transforming y to the auxiliary variable:

$$z = \ln \frac{y}{A - y} \approx \ln \frac{y}{y_{\max} - y} \tag{1}$$

since:

$$z = w_1 x_1 + w_2 x_2 + b \tag{2}$$

Measurements close to the maximum output voltage should be avoided because of the large systematic error in calculated values of z when approximating in equation (1) the unknown A with ymax. Therefore, the suitable set of measurements is along the path $(0,0)$–$(3,0)$–$(3,3)$V in the x_1, x_2-plane. Tables 2 and 3 summarize the results of measurements along the lines $(0,0)$–$(3,0)$V and $(3,0)$–$(3,3)$V respectively. The values of the variable z calculated by means of (1) are shown in the last column of the tables. Figure 3 shows the data in Table 2 in variables z and x_1 with the corresponding linear fit. The weight w_1 corresponds to the slope of the fitting line:

$$w_1 = \frac{\Delta z}{\Delta x_1} = -0.62\,\mathrm{V}^{-1} \tag{3}$$

Since $x_2 = 0.00\,\mathrm{V}$, the bias b could be estimated by crossing point of the fitting line with the z-axis:

$$b = 1.67 \tag{4}$$

$x_1(V)$	$y(V)$	z
0.00	2.31	1.68
0.26	2.24	1.50
0.59	2.15	1.29
0.83	2.08	1.15
1.09	2.00	0.99
1.32	1.92	0.85
1.72	1.76	0.59
1.98	1.65	0.41
2.20	1.57	0.29
2.53	1.44	0.10
2.88	1.29	-0.12
3.02	1.23	-0.21

Table 2: Measurements for $x_2 = 0.00\,\mathrm{V}$

$x_2(V)$	$y(V)$	z
0.00	1.23	-0.21
0.22	1.38	0.01
0.51	1.56	0.28
0.81	1.74	0.55
1.06	1.88	0.78
1.37	2.04	1.07
1.75	2.21	1.43
1.99	2.30	1.65
2.22	2.38	1.89
2.53	2.47	2.21
2.74	2.52	2.44
3.02	2.57	2.72

Table 3: Measurements for $x_1 = 3.02\,\mathrm{V}$

The data for z and x_2 in Table 3, and the corresponding linear fit are shown in Figure 4. The slope of the fitting line gives the weight w_2:

$$w_2 = \frac{\Delta z}{\Delta x_2} = 0.96\,\mathrm{V}^{-1} \tag{5}$$

According to equation (2) the crossing point, –$0,22$, of the fitting line with z axis at $x_1 = 3.02\,\mathrm{V}$ satisfies the equation

$$-0.22 = w_1 x_1 + b = -1.87 + b \tag{6}$$

Therefore, we obtain a second estimate for the bias $b = 1.65$. As a most likely estimate of b, the mean of the b-values obtained from the two graphs should be taken:

$$b = 1.66 \pm 0.01 \tag{7}$$

	Task 4	Pts
A	Linearizes the $y-x$ dependence by means of formulae (1) and (2) or equivalent	0.4
No points are awarded for the following parts B-Q if there is no documentation of a circuit that is able to provide varying voltages to the X1 and/or X2 terminals at all.		
B	Avoiding (0,3) corner (even without explicit reasoning) but award points only if two usable data-sets have been recorded	0.3
	Raw measurement values of y for varying x_1	
C	Roughly linear distribution of x_1	0.1
	Number of Points ≤ 3	0
D	4-5	0.1
	6-7	0.2
	≥ 8	0.3
	Raw measurement values of y for varying x_2	
E	Roughly linear distribution of x_2	0.1
	Number of Points ≤ 3	0
F	4-5	0.1
	6-7	0.2
	≥ 8	0.3
G	Conversion of y into z	0.2
	Plot for z versus x_1	
H	Size & Axes	0.2
I	Values in plot	0.2
J	Linear regression line (only if data is actually linear, which is not the case when plotting y vs x_1)	0.2
	Plot for z versus x_2	
K	Size & Axes	0.2
L	Values in plot	0.2
M	Linear regression line (only if data is actually linear, which is not the case when plotting y vs x_2)	0.2
N	Value for w_1 between $-0.6\,\mathrm{V}^{-1}$ and $-0.65\,\mathrm{V}^{-1}$	0.2
O	Value for w_2 between $0.93\,\mathrm{V}^{-1}$ and $1.01\,\mathrm{V}^{-1}$	0.2
P	Value for b between 1.64 and 1.70 **Note** that these boundaries are strict and outside no points are awarded here, even if they are close.	0.2
Q	Overall penalty for significant digits other than 2-3 (rare occasion only -0.2)	-0.4
	Total on Task 4	3.5

Task 5

Since X1 is connected to +V, the input voltage x must somehow be supplied through X2. Conveniently, the neuron that we have to build in this task requires a weight of $w_2/2$, which we can effectively achieve by reducing the input voltage on X2 by a factor of two. For this, one of the potentiometers can be used to halve the input voltage. To find the correct position, the resistance between terminals 1-2 and 2-3 is measured while adjusting the knob. Due to the non-linear

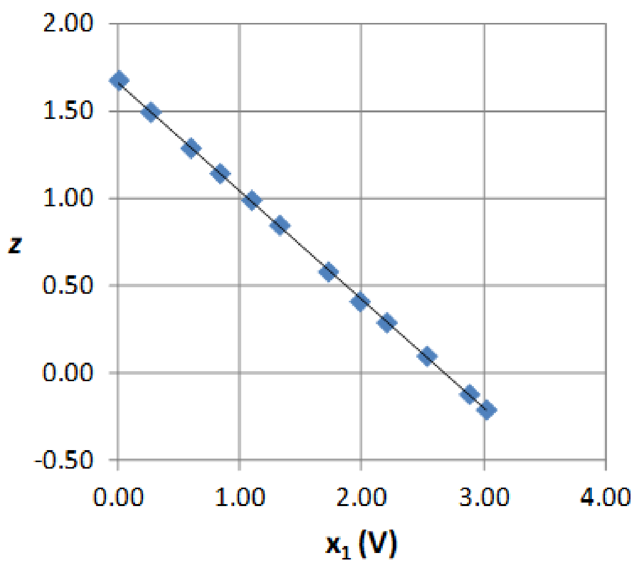

Figure 3: Graph of $x_1(V)$ vs z for Task 4

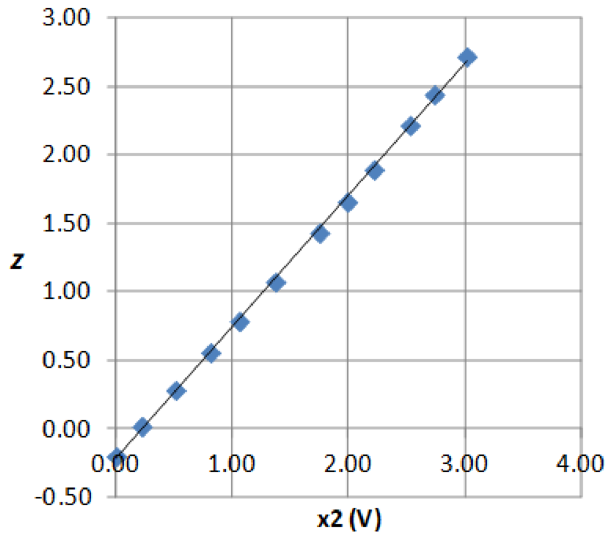

Figure 4: Graph of $x_2(V)$ vs z for Task 4

behavior of the potentiometer, the final position is not the physical middle position. Alternatively, if a voltage is applied to the potentiometer, this can be tuned with the voltmeter function. The circuit for the modified neuron is visible in Fig. 5.

For this circuit, the input voltage x is converted to x_2 via the following relation:

$$x_2 = \frac{1}{2}x$$

Therefore, the intermediate function z becomes

$$z = w_1 U_{+V} + \frac{w_2}{2}x + b.$$

This means that our new bias b_5 is the constant part, so

$$b_5 = w_1 U_{+V} + b \approx -0.35.$$

Note that it is also possible and fully correct to connect the other end of the potentiometer to +V instead of the ground, which leads to the same effective weight but a different b_5 of approximately 1.2.

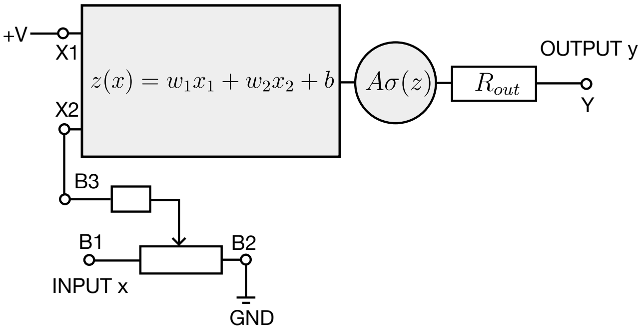

Figure 5

$x(V)$	$y(V)$	z
0.00	1.13	-0.36
0.20	1.19	-0.27
0.40	1.25	-0.18
0.60	1.32	-0.08
0.80	1.39	0.02
1.00	1.46	0.12
1.20	1.51	0.20
1.40	1.58	0.30
1.60	1.63	0.38
1.80	1.69	0.47
2.00	1.77	0.59
2.20	1.82	0.67
2.40	1.87	0.75
2.56	1.93	0.86

Table 4

However, in this solution we only consider the case shown in Fig. 5.
When the circuit is assembled, we can use the A-potentiometer (connected to +V and GND) to supply a variable voltage to the input of the new neuron. This way, we can take a few measurements to confirm that our circuit behaves as desired, which leads to the data in Tab. 4.

By linearization via the inverse activation function

$$z = -\left(\ln \frac{A}{y} - 1\right)$$

we can plot z vs x and determine the effective weight from the slope and the bias b_5 from the intercept, see Fig. 6.

Experimentally, we obtain $b_5 \approx -0.36$, which is close to the theoretical value and an effective weight of $0.47\,\mathrm{V}^{-1} \approx w_2/2$. Since the linear fit is also in good agreement with the data points, we have shown that our circuit fulfils the expectations.

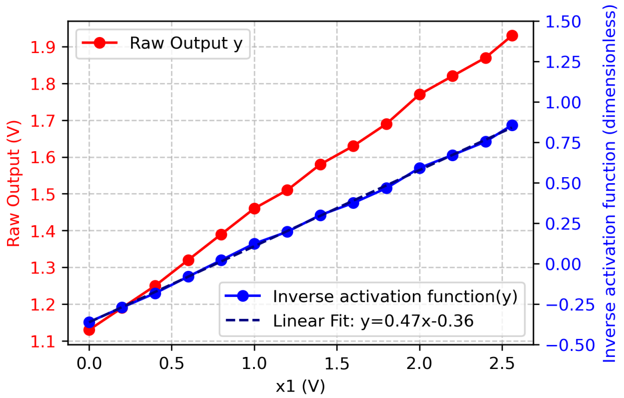

Figure 6

	Task 5	Pts
A	Idea and drawing of circuit	0.3
B	Theoretical derivation and calculation of b_5 partial credit for just the formula $b_5 = w_1 U_{+V} + b$	0.2 *0.1*
No points are awarded for the following parts C-I if the circuit used is no different than the one used in task 4 or the only change is that X1 is connected to +V, or if there is no documentation of a circuit that is able to provide varying voltages here or in previous tasks.		
 C D	Raw measurement values of y for varying x Usage of whole span for x Number of Points ≤ 2 3-4 ≥ 5	 0.1 0 0.1 0.2
 E F G	Plot for z versus x Size & Axes Converted values in plot Linear regression line	 0.2 0.2 0.2
H	Comparison of b_5	0.1
I	Overall penalty for significant digits other than 2-3	-0.2
	Total on Task 5	1.5

Task 6

a

To determine the internal series output resistance R_{out}, we have (at least) two options. We can either set the output of the neuron to a known value and connect the Y terminal via an ampere-meter directly to ground - essentially shorting it - and dividing the neuron output voltage by the measured short-circuit current.
Alternatively, we can set the neuron to a known open-circuit voltage U_{open} by connecting +V to X2, connect Y to ground via the potentiometer resistance R_{P} and measure the voltage drop U_{P} over it. This is the safer version, in case we do not know how small R_{out} is and we prevent dangerous currents. We measure:

$$U_{\text{open}} \approx 2.71\,\text{V}, \qquad U_{\text{P}} \approx 2.11\,\text{V}$$

This results in:

$$R_{\text{out}} = \left(\frac{U_{\text{open}}}{U_{\text{P}}} - 1 \right) R_{\text{P}} \approx 284\,\Omega$$

b

We want our circuit to approximate the function

$$y_6(x) = A_6 \cdot \sigma(w_2 x + b) + B_6$$

with $B_6 = 1.48\,\text{V}$. First, it is obvious that w_1 does not influence y_6, and therefore terminal X1 needs to be connected to ground. To increase the voltage of terminal Y, we need to add a voltage divider in the form of a potentiometer between terminal Y and the supply voltage, as shown in Fig. 7.

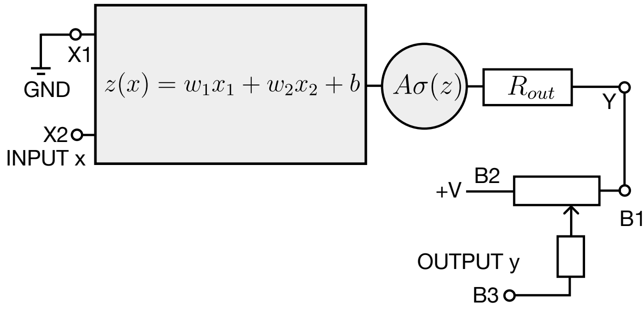

Figure 7

The output voltage can be expressed by

$$y = (1 - \eta)A\sigma(w_2x + b) + \eta U_{+V},$$

where η is the voltage division ratio of the R_P and R_{out} combined:

$$\eta = \frac{R_{out} + R_a}{R_{out} + R_P}$$

with R_a being the fraction of R_P that lies in between B1 and B3. With this, we can use the given value of B_6 to get η:

$$\eta = \frac{B_6}{U_{+V}} \approx 0.458 \Rightarrow \frac{R_a}{R_P} = \frac{1}{R_P}\left(\frac{R_{out} + R_P}{2} - R_{out}\right) \approx 0.389$$

And thus we gain a theoretical value for A_6:

$$A_6 = A(1 - \eta) = A\left(1 - \frac{B_6}{U_{+V}}\right) \approx 1.5\,\text{V}. \tag{8}$$

To experimentally verify that our neuron behaves as expected, once again, the remaining potentiometer can be used to apply different input voltages x and the output y is recorded. The multimeter can be used to set the *internal* B-potentiometer to the right ratio - do not forget to measure in the ohmmeter mode only if there are no currents running through the potentiometer. To linearize the data, we rescale the x-values by applying the sigmoid function $\sigma(z(x))$ to it. The resulting numerical values can be seen in Tab. 5.

If one plots $\sigma(w_2x+b)$ against x, one expects a linear function with slope A_6 and intercept B_6. The raw data is shown in Fig. 8 and the linear fit in Fig. 9.

$x(V)$	$y(V)$	$A\sigma(w_2x+b)(V)$
0.00	2.73	1.38
0.15	2.76	1.48
0.30	2.78	1.58
0.45	2.80	1.68
0.60	2.83	1.78
0.75	2.85	1.87
0.90	2.86	1.96
1.20	2.89	2.11
1.50	2.91	2.25
1.80	2.92	2.36
2.10	2.94	2.45
2.40	2.95	2.52
2.70	2.96	2.58
3.00	2.96	2.62
3.23	2.97	2.65

Table 5

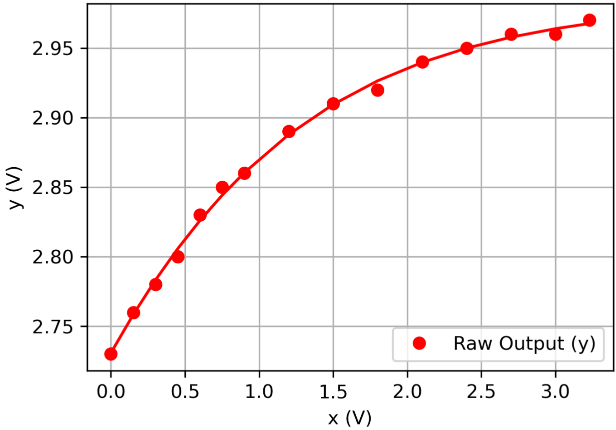

Figure 8

From the fit, we read $A_6 \approx 1.53$ and $B_6 \approx 1.45$, which is in agreement with our theoretical values up to 2 significant digits. The fact that the linear fit function agrees well with the data points further confirms that the neuron behaves as intended.

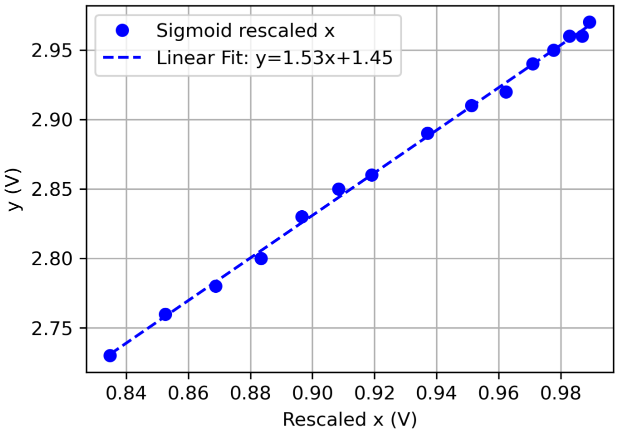

Figure 9

	Task 6	Pts
A	Idea for output resistance	0.2
B	Measurements	0.2
C	$R_{\mathrm{out}} = 284\,\Omega \pm 15\,\Omega$	0.1
D	Idea and drawing of circuit	0.5
No points are awarded for the following parts E-L if the circuit used is the same as used in task 4 or task 5 or the only change is that X1 is connected to GND or if there is no documentation of a circuit that is able to provide varying voltages here or in previous tasks.		
E	Derivation of A_6 (8)	0.4
	Penalty for neglecting R_{out}	-0.2
	Raw measurement values of y_6 for varying x	
F	Usage of whole span for x	0.1
G	Number of Points <= 2	0
	3-4	0.1
	>= 5	0.2
	Plot for y_6 versus $\sigma(z(x))$	
H	Size & Axes	0.2
I	Converted values in plot	0.2
J	Linear regression line	0.2
K	Alignment of measured and theoretical values	0.2
L	Overall penalty for significant digits other than 2-3	-0.2
	Total on Task 6	2.5