E2 - Solution

Task a.

The light transmitted through the film forms a primary and secondary diffraction pattern as displayed in Fig. 13.

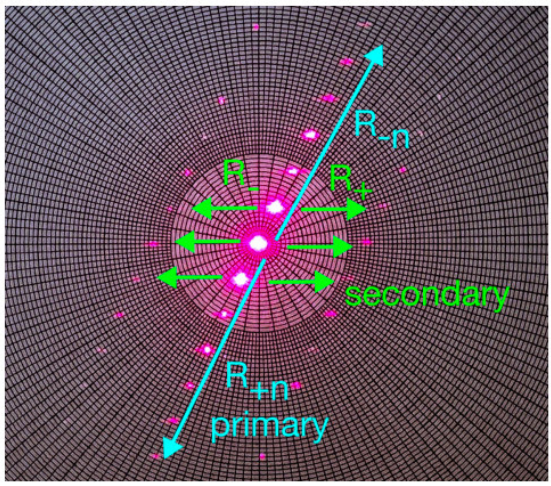


Figure 13: Photo of the diffraction pattern with the primary and secondary pattern marked.

The secondary diffraction pattern is only required for task d. For the primary diffraction pattern, a number of bright interference maxima lying on the same line is visible. This fact implies that locally, within the cross section of the laser spot, the printed lines on the film surface form a diffraction grating consisting of a large number of practically linear parallel fringes. Therefore, the diffraction maxima are situated on a straight line, perpendicular to the tangent to the illuminated sinusoids, like illustrated in Fig. 14.

By scanning the laser in a horizontal direction, the diffraction pattern will tilt according to the line slope in the illuminated spot. The pattern will be vertical when the laser incidents on crests or valleys of the illuminated sinusoids. Therefore, the distance between two consecutive vertical positions of the diffraction pattern is $l = \Lambda/2$, hence $\Lambda = 2l$.

The experimental setup is designed as follows: As a first step, we fix the foil with two clips to the L-shaped stand and align it carefully vertically. The goniometer is glued to the screen with the 0°-division pointing vertically. Next we fix the laser to another L-shaped stand and align it so that the beam hits the centre of the goniometer. We place the screen (goniometer) as far as possible from the foil to achieve larger displacements of the maxima and hence, a better precision, also see Fig. 15. The stand with the foil is being displaced in small steps across the laser beam, and the angle of inclination θ of the diffraction pattern is being measured as function of the distance x between the laser spot and the edge of the film.

Further we put the ruler on the diagonal scale so that one of its edges is aligned with the alignment line on the scale; we'll be sliding the L-shaped stand with the foil along the edge of the ruler. That way we can focus on observing how the diffraction maxima shift while we slide the stand, without a need for sharing our attention between the diffraction maxima and alignment of the stand. A sample data set is recorded in the first two columns of Table 7, while the corresponding graph of θ vs. x is shown in Fig. 16.

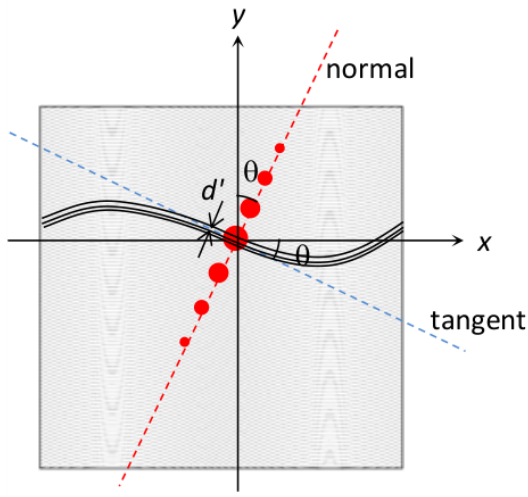


Figure 14: Qualitative sketch of interference pattern.



Figure 15: Setup with optimized usage of the table space for improved precision.

In what follows we'll be outlining two possible approaches for achieving precise experimental results of the quantities asked in this problem: **approach A**: graphical; **approach B**: carefully scanning the diffraction pattern around critical configurations.

Approach A. Points x_1 and x_2 in Fig. 16 corresponding to $\theta = 0°$, i.e. consecutive crest and valley of the sinusoid, can be obtained by linear fits of the four points nearest to the zero-crossing points:

$$x_1 = (10.0 \pm 0.5)\,\text{mm} \qquad x_2 = (40.5 \pm 0.5)\,\text{mm}\,.$$

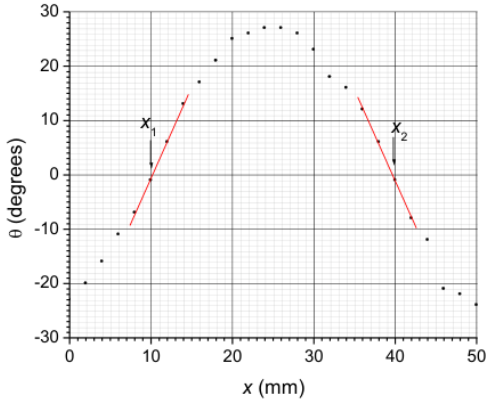


Figure 16: Graph of θ vs. x.

Therefore, $l = x_2 - x_1 = 30.5\,\text{mm}$ with uncertainty $\delta l = \sqrt{\delta x_1^2 + \delta x_2^2} = 0.7\,\text{mm}$, and

$$\Lambda = (61.0 \pm 1.4)\,\text{mm}\,.$$

Approach B. Alternatively, we move the stand with the foil so that the diffraction maxima align along a vertical line and take the reading x_1 from the diagonal scale. Next we slide the stand along the rule to find the other position where the diffraction maxima align along a vertical line and take the corresponding reading x_2. We can see that achieving a vertical alignment is not easy and incurs an uncertainty, so we need to repeat the measurements. The measurement data are shown in the table below.

No	x_1/mm	x_2/mm
1	77.8	47.3
2	78.0	47.4
3	77.7	47.5
4	77.6	47.3
5	77.7	47.4
avg	77.76	47.38

According to this we calculate

$$\Lambda = 2(x_2 - x_1) = 60.8\,\text{mm}\,.$$

and $\Delta\Lambda = 0.16\,\text{mm}$

	Task E2.a	Pts
A	Idea for linking the changing tilt of the primary diffraction pattern to phase of sinusoid.	0.3
	Partial credit if full Λ is measured between consecutive vertical positions of the diffraction pattern	*0.1*
B	Sketch of the correct setup (shifting the laser laterally to scan foil position and screen behind)	0.2
	partial credit if $\Lambda/2$ is measured between two "maximal" inclinations	*0.1*
C	Using the diagonal scale for measuring x (data recorded with the precision of 0.1 mm)	0.2
	If Approach A is chosen:	
D	Usage of at least half the sinusoid period (30.5 mm) for variation of x	0.2
E	Number of recorded points:	
	≥ 6 but < 10 points recorded	*0.2*
	≥ 10 points recorded	0.3
	If Approach B is chosen:	
E!	Number of measurements n for both x_1 and x_2: $0.1(n-2)$, in total up to	0.3
D!	Quality of data: standard deviation between individual measurements of $\lvert x_1 - x_2\rvert \leq 0.2$ mm	0.2
	$\lvert x_1 - x_2\rvert \leq 0.3$ mm	*0.1*
F	Distance between the foil and the screen at least 60 cm	0.2
G	Partial points for value of Λ between 58 mm and 64 mm	*0.1*
	Partial points for value of Λ between 59.5 mm and 62.5 mm	*0.2*
	between 60 mm and 62 mm	*0.3*
	Full points: between 60.5 mm and 61.5 mm	0.4
H	Suitable error estimation	0.2
	If the error estimation is based on a reasonable approach, but the numerical estimates of the direct measurement uncertainties are clearly under- or overestimated	*0.1*
	If the error estimation method itself is flawed or not provided	*0*
	Total on Task E2.a	2.0

Task b.

Approach A. It can be easily deduced from Fig. 14 that the period of the diffraction grating, perpendicularly to the printed lines, is

$$d' = d\cos\theta\,.$$

The angle to the n-th order diffraction maximum is:

$$\varphi = \sin^{-1}\left(\frac{n\lambda}{d\cos\theta}\right).$$

The distance between the 0-th and the n-th maxima on the screen is

$$R_n = L\tan\varphi \approx \frac{n\lambda L}{d\cos\theta}\,, \tag{9}$$

where $L = 74.6$ cm is the distance between the film and the screen, and $\sin\varphi \approx \tan\varphi$ since involved angles are much smaller than 1 rad. Therefore, if we choose a specific order maximum and measure the distance R at different points on the screen, i.e. for different angles θ, the distance d can be calculated as a sample average:

$$d = n\lambda L \left\langle \frac{1}{R\cos\theta} \right\rangle$$

While the uncertainty –as a sample standard deviation of d. Since the 0-th order could be slightly offset from the goniometer center, we measure the corresponding distances between the two symmetric, n-th and $-n$-th, maxima: $D_n = R_n + R_{-n}$ and calculate

$$d = 2\lambda L \left\langle \frac{1}{D_n\cos\theta} \right\rangle$$

The last four columns of Table 7 summarize the measured distances and calculated value of d for the 5-th maximum. By averaging the d values, we obtain:

$$d = (60.1 \pm 0.5)\,\mu\text{m}\,.$$

However, we need to keep in mind the uncertainty of the laser wave length. Adding the relative errors of the measurement data and laser wavelength according to the Pythagorean rule (applicable for uncorrelated error sources), we obtain

$$d = (60.1 \pm 0.7)\,\mu\text{m}\,.$$

We note that if we use the data at $\theta = 0°$, the diffraction maxima yield directly the line distance d according to

$$d = 2\lambda/\sin\alpha_n = 2\lambda\sqrt{L^2 + D_n^2/4}/D_n. \tag{10}$$

Approach B. Alternatively to using many data points for different θ, we can choose the positions $x = x_1$ or $x = x_2$ found in the previous task since they are the lateraly points offering the highest precision. Analogously to approach A, we determine the distance between the symmetric diffraction maxima of highest observable order, $n = 6$, to achieve the highest possible precision. The best way to determine the distance D_n is by marking dots onto the screen at the positions of the maxima, and measure the distance between the dots by ruler (to keep the goniometer clean, one can attach another sheet of paper to the stand). Since a single measurement will be very precise if done carefully, repeated measurements are not required for this task. The result of the measurement is $D_6 = 105.5\,\text{mm}$ with $L = 810\,\text{mm}$, resulting in $d = 59.7\,\mu\text{m}$. Estimated error is $\pm 0.7\,\mu\text{m}$.

	Task E2.b	Pts
A	Understanding that the primary diffraction pattern is created by the distance of the sinusoids to each other	0.2
B	Using ≥ 5, ≥ 7, ≥ 10 (θ, D_n) data points to receive 0.1, 0.2, 0.4 pts. (*approach A*) or chossing to record data at $x = x_1$ or $x = x_2$ (*approach B*) if vertical interference pattern is used without an explanation (*approach B*)	0.4 *0.2*
C	Expression equivalent to Eq. 10 simplified Eq. 10 (without Pythagorean correction)	0.3 *0.2*
D	Usage of at least a total span of 6+6=12 diffraction orders for measurement of ϕ Total span from 9 to 11 diffraction orders for measurement of ϕ Total span from 6 to 8 diffraction orders for measurement of ϕ If measurement data is not consistent with real experiment	0.3 *0.2* *0.1* *0*
E	Distance to the screen at least $L \geq 70\,\mathrm{cm}$ Partial credit for L between 40 cm and 70 cm L between 20 cm and 40 cm	0.3 *0.2* *0.1*
F	Partial points for value of d between 58 µm and 62 µm between 58.5 µm and 61.5 µm Full points for value of d between 59 µm and 61 µm	*0.1* *0.2* 0.3
G	Suitable error estimation If the error estimation is based on a reasonable approach, but the numerical estimates of the direct measurement uncertainties are clearly under- or overestimated If the error estimation method itself is flawed or not provided	0.2 *0.1* *0*
	Total on Task E2.b	2.0

Task c.

Approach A: The point where a sinusoid crosses the x-axis corresponds to a maximum angle of inclination of the diffraction pattern $\theta = (27 \pm 1)^\circ$ and can be obtained as:

$$x_0 = \frac{x_1 + x_2}{2} = (25.3 \pm 0.7)\,\mathrm{mm}\,.$$

The sinusoid equation can be written as

$$y = A \sin\left(k(x - x_0)\right)$$

where:

$$k = \frac{2\pi}{\Lambda} = (0.103 \pm 0.002)\,\mathrm{mm}^{-1}$$

is the sinusoid wavevector. Since $\tan\theta = \mathrm{d}y/\mathrm{d}x$, we obtain:

$$\tan\theta = kA\cos\left(k(x - x_0)\right).$$

Therefore, the auxiliary variables $z = \cos(k(x - x_0))$ and $t = \tan\theta$ are related by a linear dependence

$$t = kAz \equiv mz \tag{11}$$

and the amplitude of the sinusoid can be calculated by determining the slope coefficient m:

$$A = \frac{m}{k}\,.$$

Calculated values of t and z are shown in the third and fourth column of Table 7, and the corresponding graph is shown in Fig. 17.

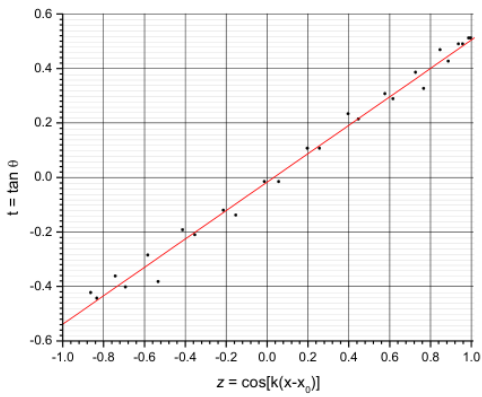


Figure 17: Graph of $\tan\theta$ vs. $\cos(k(x - x_0))$.

The slope coefficient is

$$m = (0.52 \pm 0.01)\,.$$

Therefore:

$$A = (5.0 \pm 0.2)\,\mathrm{mm}\,.$$

Approach B. The amplitude of the sinusoids on the foil can be determined if we know the maximal slope the printed lines $\max_x \frac{\mathrm{d}y}{\mathrm{d}x} = Ak$ as we already know the wave vector $k = 2\pi/\Lambda$. We notice that the diffraction maxima lay on a line perpendicular to the lines on the foil at the point where the laser beam passes it, hence we can find the maximal slope of the lines as the maximal angle θ that the array of diffraction maxima form with the vertical axis,

$$\max_x \frac{\mathrm{d}y}{\mathrm{d}x} = \max \tan\theta = \tan\theta_{\max} \tag{12}$$

This is done easily by using the same setup as in part A, by sliding the stand of the foil along the edge of the ruler and observing how the maxima move on the screen. As a result we obtain $\theta_{\max} = (27.5 \pm 0.3)^\circ$, corresponding to

$$A = \Lambda \tan\theta_{\max}/2\pi = (5.05 \pm 0.06)\,\mathrm{mm}\,.$$

	Task E2.c	Pts
A	Recognition that the slope of the sinusoid is perpendicular to the primary diffraction pattern	0.3
B	Idea to use that the first derivative of the sinusoid is $\tan\theta$	0.3
C	Correct linearization equivalent to Eq. 11 (approach A) or deriving Eq. (12) (approach B)	0.5
	If approach A has been chosen	
D	Computing the auxiliary (linearized) value-pairs (z and t) for	
	≤ 5 points	*0.0*
	6-9 points	*0.2*
	≥ 10 points	0.4
	Plot for t versus z	
E	Suitable graphical evaluation to find A (approach A)	0.2
	If approach B has been chosen	
E!	Idea of using the maximal inclination point	0.2
D!	Obtaining $\theta_{\max}$	
	Partial points within $\theta_{\max} = (27.5 \pm 1.0)°$	*0.2*
	Full points within $\theta_{\max} = (27.5 \pm 0.5)°$	0.4
F	Distance to the screen at least $L \geq 70\,\text{cm}$	0.3
	Partial credit for L between 40 cm and 70 cm	*0.2*
	L between 20 cm and 40 cm	*0.1*
G	Partial points for value of A between 4.5 mm and 5.5 mm	0.2
	Partial points for value of A between 4.7 mm and 5.3 mm	0.5
	Full points for value of A between 4.8 mm and 5.2 mm	0.8
H	Suitable error estimation	0.2
	If the error estimation is based on a reasonable approach, but the numerical estimates of the direct measurement uncertainties are clearly under- or overestimated	*0.1*
	If the error estimation method itself is flawed or not provided	*0*
	Total on Task E2.c	3.0

Task d.

The vertical slabs that come from the printing technique lead to the secondary diffraction pattern as can be seen from the photo in Fig. 13. In this task, there also exist two different approaches that differ in the way the diffraction vectors are modelled but lead to the same and valid result.

Approach A: In the model of Huygen's elementary waves, the elementary wave sources along each of the vertical edges, as shown in thick red dotted lines in Fig. 18, will create wavefronts that propagate towards the screen and create interference pattern.

These edges have a regular distance g of each other that depends on θ, specifically:

$$g = \frac{s}{\tan\theta}\,.$$

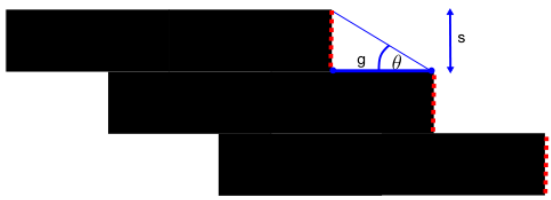


Figure 18: Sketch for explanation of the relevant quantities for secondary diffraction pattern

θ	$\tan(\theta)$	$R_{+1} + R_{-1}$ (cm)	s (μm)
27.8	0.527	4.75	1.08E-05
25.7	0.481	4.65	1.01E-05
24.5	0.456	4.50	9.88E-06
24.7	0.460	4.60	9.76E-06
21.8	0.400	3.90	1.00E-05
18.1	0.327	3.00	1.06E-05

Table 6: Measurements of the secondary diffraction order distances for different angles θ

From the diffraction angle ω of the maxima of the secondary diffraction pattern, g can be expressed via:

$$\frac{\lambda}{g} = \sin\omega \approx \omega\,.$$

Experimentally, we can obtain ω via

$$\omega \approx \tan\omega = \frac{R_+ + R_-}{2L}\,,$$

where $R_+ + R_-$ is the distance between the two diffraction orders visible (left and right from the centre) in the secondary pattern. We can combine this knowledge to obtain g:

$$g = \frac{2L\lambda}{R_+ + R_-}$$

Thus, we get s via the trigonometric relation

$$s = g\tan\theta = \frac{2L\lambda\tan\theta}{R_+ + R_-} \tag{13}$$

It is important to notice that the secondary pattern can only be observed distinctly for large θ since only in these regions, the slope of the sinusoid barely changes, which in turn means that θ and the resulting diffraction angle ω is rather constant. In Tab. 6, the recorded measurement points for the same $L = 74.6\,\text{cm}$ is shown.

A graphical evaluation has no benefit over computing the average of the point-wise results here, so we use the average of s as the result and its standard deviation as the error estimate. Thus, we get:

$$s \approx (10.2 \pm 0.4)\,\mu\text{m}\,.$$

Approach B: Alternatively the characteristic diffraction length is the hypotenuse of length h of the blue triangle shown in Fig. 18, which is $h = s/\sin\theta$. Subsequently, the diffraction angle is

$$\omega \approx \sin\omega = \frac{\lambda}{h} = \frac{\tilde{R}_+ + \tilde{R}_-}{2L}\,,$$

Where $\tilde{R}_+$ and $\tilde{R}_-$ are the positions of the secondary diffraction pattern orthogonal to the primary pattern. Thus, the formula for s becomes

$$s = \frac{2L\lambda\sin\theta}{\tilde{R}_+ + \tilde{R}_-}\,, \tag{14}$$

which leads to the same outcome as in approach A since $\tilde{R}_+ = \cos\theta R_+$ and $\tilde{R}_- = \cos\theta R_-$.

	Task E2.d	Pts
A	Linking the horizontal line segments of the discrete printer resolution to the secondary diffraction pattern	0.3
B	Understanding that the diffraction angle ω depends on θ	0.3
C	Deriving the final formula to compute s as in Eq. 13, or, alternatively Eq. 14	0.4
D	Sketching or describing a suitable setup and procedure to measure the relevant quantities to determine s	0.3
E	Using both the plus and minus diffraction order for improved measurement precision	0.2
F	Method of measuring R_+ and R_-: Marking the secondary diffraction maxima with pen on screen and linear regression - evidence either via screen paper that shows this method or concise description of this	0.3
	Making just a single measurement between two horizontal interference points	*0.1*
G	Distance to the screen at least $L \geq 70\,\text{cm}$	0.3
	Partial credit for L between 40 cm and 70 cm	*0.2*
	L between 20 cm and 40 cm	*0.1*
H	Choosing $\theta > 27°$	0.2
	partial credit for $\theta > 25°$	*0.1*
I	Partial points for value of s between 8 µm and 12 µm	0.1
	Partial points for value of s between 9 µm and 11 µm	0.3
	Full points for value of s between 9.4 µm and 10.6 µm	0.5
J	Suitable error estimation	0.2
	If the error estimation is based on a reasonable approach, but the numerical estimates of the direct measurement uncertainties are clearly under- or overestimated	*0.1*
	If the error estimation method itself is flawed or not provided	*0*
	Total on Task E2.d	3.0

Table 7: Measurement data for experiment “hidden pattern”

x (mm)	θ (°)	z	t	R_5 (cm)	R_{-5} (cm)	D_5 (cm)	d (cm)
0	-23	-0.86	-0.42447	4.4	4.4	8.8	0.00602
2	-20	-0.74	-0.36397	4.4	4.3	8.7	0.00597
4	-16	-0.58	-0.28675	4.2	4.2	8.4	0.00604
6	-11	-0.41	-0.19438	4.1	4.1	8.2	0.00606
8	-7	-0.21	-0.12278	4.1	4.1	8.2	0.00599
10	-1	-0.01	-0.01746	4.1	4.1	8.2	0.00595
12	6	0.20	0.105104	4.1	4.1	8.2	0.00598
14	13	0.40	0.230868	4.2	4.2	8.4	0.00596
16	17	0.58	0.305731	4.3	4.3	8.6	0.00593
18	21	0.73	0.383864	4.3	4.3	8.6	0.00608
20	25	0.85	0.466308	4.4	4.4	8.8	0.00612
22	26	0.94	0.487733	4.5	4.6	9.1	0.00597
24	27	0.99	0.509525	4.6	4.5	9.1	0.00602
26	27	1.00	0.509525	4.6	4.5	9.1	0.00602
28	26	0.96	0.487733	4.5	4.6	9.1	0.00597
30	23	0.89	0.424475	4.4	4.4	8.8	0.00602
32	18	0.77	0.32492	4.2	4.2	8.4	0.00611
34	16	0.62	0.286745	4.2	4.3	8.5	0.00597
36	12	0.45	0.212557	4.1	4.1	8.2	0.00608
38	6	0.26	0.105104	4.1	4.1	8.2	0.00598
40	-1	0.06	-0.01746	4.1	4.1	8.2	0.00595
42	-8	-0.15	-0.14054	4.1	4.1	8.2	0.00601
44	-12	-0.35	-0.21256	4.2	4.1	8.3	0.00601
46	-21	-0.53	-0.38386	4.4	4.4	8.8	0.00594
48	-22	-0.69	-0.40403	4.4	4.4	8.8	0.00598
50	-24	-0.83	-0.44523	4.4	4.4	8.8	0.00607