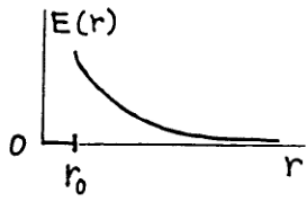
# Theoretical Problem 3 -- Solution

1. By symmetry, the electric field will point radially away from the wire, and its magnitude will depend only on the radius $r$ (in cylindrical coordinates). Place an imaginary cylinder around the wire and use Gauss's law:

$$2\pi r E(r) = \frac{q_{\text{linear}}}{\varepsilon_0}$$

for a cylinder of radius $r$ and unit length, provided $r \geq r_0$. Therefore

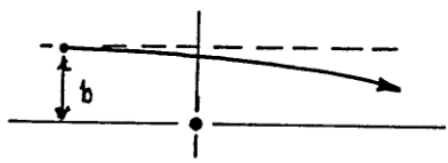
$$E(r) = \frac{q_{\text{linear}}}{2\pi r \varepsilon_0} = \frac{0.791}{r}\, N/C \quad \text{provided} \quad r \geq r_0.$$




When $r < r_0$, the electric field is zero (because copper is a good conductor), that is, the electric field is zero inside the wire.

2. The problem stated that the angular deflection is small. Estimate the deflection angle $\theta_{\text{final}}$ by forming a quotient: the momentum acquired transverse to the initial velocity divided by the initial momentum:

$$\theta_{\text{final}} \cong \frac{|\Delta \mathbf{p}_\perp|}{m v_0}.$$




A first estimate of the transverse momentum can be made as follows:

The transverse force (where it is significant) is of order $\dfrac{e q_{\text{linear}}}{2\pi\varepsilon_0 b}$.

The (significant) transverse force operates for a time such that the electron goes a distance of order $2b$, and hence that transverse force operates for a time of order $2b/v_0$.

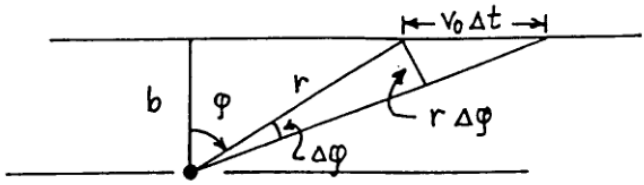
The product of force and operating time gives an estimate the transverse momentum:

$$|\Delta \mathbf{p}_\perp| \cong \frac{e q_{\text{linear}}\, 2b}{2\pi\varepsilon_0 b v_0} = \frac{e q_{\text{linear}}}{\pi \varepsilon_0 v_0},$$

and so $\theta_{\text{final}} \cong \dfrac{e q_{\text{linear}}}{\pi\varepsilon_0 m v_0^2} = \dfrac{q_{\text{linear}}}{\pi\varepsilon_0 2V_0} = 3.96 \times 10^{-5}$ radians

after one uses energy conservation to say $\frac{1}{2} m v_0^2 = e V_0$. Note that the deflection is extremely small and that the deflection is independent of the impact parameter $b$. Because the force between the positively charged wire and the electron is attractive, the deflection will bend the trajectory toward the wire--though only ever so slightly.

A more accurate estimate can be made by setting up an elementary integration for $|\Delta \mathbf{p}_\perp|$, as follows. For the sake of the integration, approximate the actual trajectory by a straight line that passes the wire at distance $b$, as shown in the sketch.




$$|\mathbf{F}_\perp| = \frac{eq_{\text{linear}}}{2\pi\varepsilon_0 r}\cos\varphi \qquad v_0\,\Delta t\cos\varphi = r\Delta\varphi \quad \text{and so} \quad \Delta t = \frac{r\Delta\varphi}{v_0\cos\varphi}$$

$$|\mathbf{F}_\perp|\Delta t = \frac{eq_{\text{linear}}}{2\pi\varepsilon_0 r}\cos\varphi\frac{r\Delta\varphi}{v_0\cos\varphi} = \frac{eq_{\text{linear}}}{2\pi\varepsilon_0 v_0}\Delta\varphi.$$

Adding up the increments in $\Delta\varphi$ over the range $-\pi/2$ to $\pi/2$ yields $|\Delta\mathbf{p}_\perp| = \dfrac{eq_{\text{linear}}}{2\varepsilon_0 v_0}$.

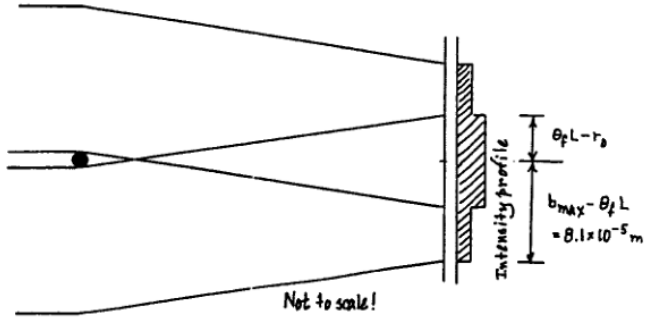
The better estimate differs from the first estimate by merely the factor $\pi/2$. The better estimate yields

$$\theta_{\text{final}} \cong \frac{eq_{\text{linear}}}{2\varepsilon_0 m v_0^2} = \frac{q_{\text{linear}}}{2\varepsilon_0 2V_0} = 6.21\times10^{-5}\ \text{radians}.$$

3. Most of the bending of the trajectory occurs within a distance from the wire of order $b$. On the scale of $L$, order $b$ is very small indeed. Therefore we may approximate the trajectory by two straight lines with a kink near the wire. Thus, at the viewing surface, the transverse displacement of each trajectory is

$$\begin{pmatrix}\text{transverse}\\ \text{displacement}\end{pmatrix} = \theta_{\text{final}}L = 6.21\times10^{-5}\times0.3 = 1.86\times10^{-5}\ \text{meter} \cong 19\,r_0 >> r_0.$$

Thus the portions of the beam that pass on opposite sides of the wire have a region of overlap, as shown in the sketch.




The full width of the overlap region is

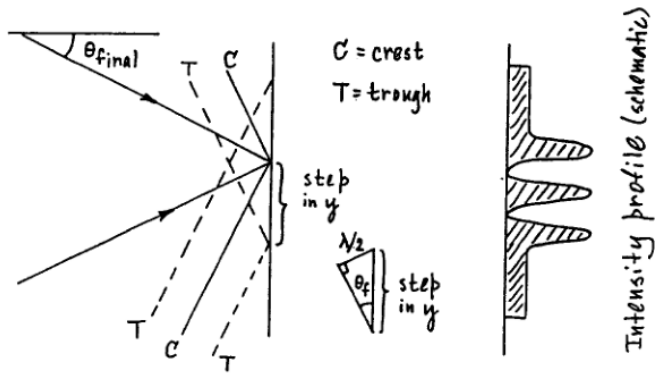
$$\begin{pmatrix}\text{full}\\ \text{width}\end{pmatrix} = 2\times\left(\theta_{\text{final}}L - r_0\right) \cong 36\,r_0 = 36\times10^{-6} \quad \text{meter}.$$

The density of impacts is constant within each region and doubled in the overlap region.

4. Associated with the electron beam is a quantum wave pattern whose de Broglie wavelength is

$$\lambda = \frac{h}{mv_0} = \frac{h}{\sqrt{2meV_0}} = 8.68\times10^{-12} \quad \text{meter}.$$

The de Broglie wavelength is so much smaller than the beam width $2b_{\max}$ that one may ignore "single slit diffraction" effects. Rather, to the right of the wire, two plane waves that travel at a fixed angle relative to each other (an angle $2\theta_{\text{final}}$) overlap and interfere. In the region where, classically, the two halves of the original beam overlap, there will be interference maxima and minima.




Reference to the sketch indicates that

$$\begin{pmatrix}\text{Interval between}\\ \text{adjacent constructive}\\ \text{interference locations}\end{pmatrix} = \begin{pmatrix}\text{step}\\ \text{in } y\end{pmatrix} = \frac{\lambda/2}{\sin\theta_{\text{final}}} \cong \frac{\lambda/2}{\theta_{\text{final}}} \cong \frac{\frac{1}{2}\times 8.68\times 10^{-12}}{6.21\times 10^{-5}} = 7.00\times 10^{-8}$$

Because the region of overlap has a full width of $\cong 36\times 10^{-6}$ meter, there will be roughly 500 interference maxima. Note that the interval between adjacent maxima does *not* depend on either $b$ or $b_{\max}$ (unlike the situation with ordinary "double slit interference").

**Historical note.** This problem is based on the now-classic experiment by G. Mollenstedt and H. Duker, "Observation and Measurement of Biprism Interference with Electron Waves," Zeitschrift fur Physik, **145**, pp. 377-397 (1956).

Theoretical Problem 3: Grading Scheme

Part 1. 1 point.

E(r) correct outside of wire: 1 point.

E(r) inside wire: ignore in the grading. (Some students may ignore the interior because there is no field there.)

Part 2. 5 points, distributed as follows:

$\theta_{\text{final}}$ independent of $b$: 1 pt.

$\theta_{\text{final}} \propto \dfrac{eq_{\text{linear}}}{\varepsilon_0 m v_0^2}$ or $\dfrac{q_{\text{linear}}}{\varepsilon_0 V_0}$ or equivalent: + 1 pt.

Numerical coefficient correct to within a factor of 4: + 2 pts.

Numerical coefficient correct to within 20 %: + 1 pt.

Part 3: 1.5 points:

Overlap region exists: 0.5 pt.

Constant densities of impacts within each region: + 0.25 pt.

Correct ratio of intensities: + 0.25 pt.

Full width of pattern correct, given student's value for $\theta_{\text{final}}$: + 0.25 pt.

Width of overlap region correct, given student's value for $\theta_{\text{final}}$: + 0.25 pt.

Part 4: 2.5 points:

Recognizes that "two wave" interference occurs: 0.5 pt.

Correct de Broglie wavelength : 0.5 pt.

Correct separation of maxima: + 1.5 pts.

[If separation of maxima is wrong by merely a factor of 2, then partial credit: +1 pt.]

Maxima in intensity = 4 times single-wave intensity: ignore in grading.