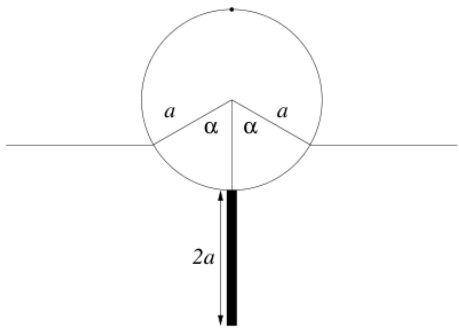
Solutions to Theoretical Question 3

(a)

The mass of the rod is given equal to the mass of the cylinder M which itself is $\pi a^2 l d$. Thus the total mass equals $2M = 2\pi a^2 l d$. The mass of the displaced water is surely less than $\pi a^2 l \rho$ (when the buoy is on the verge of sinking). Using Archimedes' principle, we may at the very least expect that

$$2\pi a^2 l d < \pi a^2 l \rho \quad \text{or} \quad d < \rho/2$$

In fact, with the floating angle $\alpha\,(< \pi)$ as drawn, the volume of displaced water is obtained by geometry:



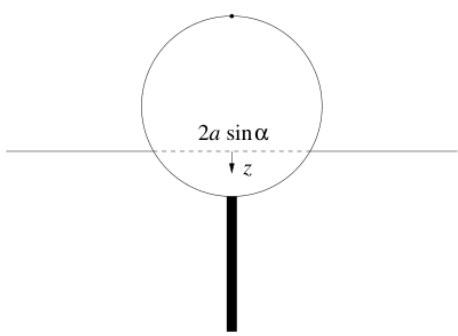
$$V = la^2\alpha - la^2 \sin\alpha \cos\alpha \ .$$

By Archimedes' principle, the mass of the buoy equals the mass of displaced water. Therefore, $2\pi a^2 l d = la^2\rho(\alpha - \sin\alpha\cos\alpha)$, i.e. α is determined by the relation

$$\alpha - \sin\alpha\cos\alpha = 2d\pi/\rho \ .$$

(b)

If the cylinder is depressed a *small* distance z vertically from equilibrium, the nett upward restoring force is the weight of the extra water displaced or $g\rho.2a\sin\alpha.lz$, directed oppositely to z. This is characteristic of simple harmonic motion and hence the Newtonian equation of motion of the buoy is (upon taking account of the extra factor 1/3)



$$8M\ddot{z}/3 = -2\rho g l z a \sin\alpha \quad \text{or} \quad \ddot{z} + \frac{3\rho g \sin\alpha}{4\pi d a} z = 0 \quad ,$$

and this is the standard sinusoidal oscillator equation (like a simple pendulum). The solution is of the type $z = \sin(\omega_z t)$, with the angular frequency

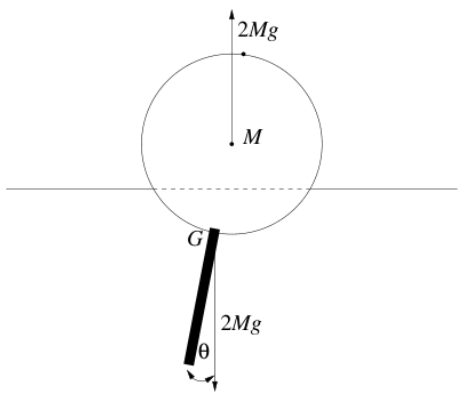
$$\omega_z = \sqrt{\frac{3\rho g \sin\alpha}{4\pi d a}} = \sqrt{\frac{3g\sin\alpha}{2a(\alpha - \cos\alpha\sin\alpha)}} \ ,$$

where we have used the relation worked out at the end of the first part.

(c)

Without regard to the torque and only paying heed to vertical forces, if the buoy is swung by some angle so that its weight is supported by the nett pressure of the water outside, the volume of water displaced is the same as in equilibrium. Thus the centre of buoyancy remains at the same distance from the centre of the cylinder. Consequently we deduce that the buoyancy arc is an arc of a circle centred at the middle of the cylinder. In other words, *the metacentre M of the swinging motion is just the centre of the cylinder*. In fact the question assumes this.

We should also notice that the centre of mass G of the buoy is at the point where the rod touches the cylinder, since the masses of rod and cylinder each equal M. Of course the cylinder will experience a nett torque when the rod is inclined to the vertical. To find the period of swing, we first need to determine the moment of inertia of the solid cylinder about the central axis; this is just like a disc about the centre. Thus if M is the cylinder mass



$$I_0 = Ma^2/2 \left(= \int_0^a r^2\, dm = \int_0^a r^2 . 2Mr\, dr/a \right)$$

The next step is to find the moment of inertia of the rod about its middle,

$$I_{rod} = \int_{-a}^{a} (Mdx/2a).x^2 = [Mx^3/6a]_{-a}^{a} = Ma^2/3 \ .$$

Finally, use the parallel axis theorem to find the moment of inertia of the buoy (cylinder + rod) about the metacentre M,

$$I_M = Ma^2/2 + [Ma^2/3 + M(2a)^2] = 29Ma^2/6 \ .$$

(In this part we are neglecting the small horizontal motion of the bentre of mass; the water is the only agent which can supply this force!) When the buoy swings by an angle θ about equilibrium the restoring torque is $2Mga\sin\theta \simeq 2Mga\theta$ for small angles, which represents simple harmonic motion (like simple pendulum). Therefore the Newtonian rotational equation of motion is

$$I_M\ddot{\theta} \simeq -2Mga\theta \ , \quad \text{or} \quad \ddot{\theta} + \frac{12g}{29a} = 0 \ .$$

The solution is a sinusoidal function, $\theta \propto \sin(\omega_\theta t)$, with angular frequency

$$\omega_\theta = \sqrt{12g/29a} \ .$$

(d)

The accelerometer measurements give

$$T_\theta/T_z \simeq 1.5 \quad \text{or} \quad (\omega_z/\omega_\theta)^2 \simeq 9/4 \simeq 2.25 \ . \quad \text{Hence}$$

$$2.25 = \frac{3g\sin\alpha}{2a(\alpha - \sin\alpha\cos\alpha)} \frac{29a}{12g} \ ,$$

producing the (transcendental) equation

$$\alpha - \sin\alpha\cos\alpha \simeq 1.61\sin\alpha \ .$$

Since 1.61 is not far from 1.57 we have discovered that a physically acceptable solution is $\alpha \simeq \pi/2$, which was to be shown. (In fact a more accurate solution to the above transcendental equation can be found numerically to be $\alpha = 1.591$.) Setting $alpha = \pi/2$ hereafter, to simplify the algebra, $\omega_z^2 = 3g/\pi a$ and $4d/\rho = 1$ to a good approximation. Since the vertical period is 1.0 sec,

$$1.0 = (2\pi/\omega_z)^2 = 4\pi^3 a/3g \quad ,$$

giving the radius $a = 3 \times 9.8/4\pi^3 = .237$ m.
We can now work out the mass of the buoy (in SI units),

$$2M = 2\pi a^2 ld = 2\pi a^2 a.\rho/4 = \pi a^3\rho/2 = \pi \times 500 \times (.237)^3 \simeq 20.9 \text{ kg} \ .$$