Question 1

A *Bungee Jumper*

(a) The jumper comes to rest when

$$\text{lost gravitational potential energy} = \text{stored strain energy}$$

$$mgy = \tfrac{1}{2}k(y-L)^2$$

0.1

$$ky^2 - 2y(kL+mg) + kL^2 = 0$$

0.1

This is solved as a quadratic.

$$y = \frac{2(kL+mg) \pm \sqrt{4(kL+mg)^2 - 4k^2L^2}}{2k}$$

$$= \frac{kL+mg \pm \sqrt{2mgkL + m^2g^2}}{k}$$

0.2

Need positive root; lower position of rest (other root after initial rise). 0.1

0.5

(b) The maximum speed is attained when the acceleration is zero and forces balance; i.e. when $mg = kx$ 0.1

Also kinetic energy = lost potential energy – strain energy within elastic rope 0.1

$$\tfrac{1}{2}mv^2 = mg(L+x) - \tfrac{1}{2}kx^2$$

0.1

$$x = \frac{mg}{k}$$

$$v^2 = 2g\left(L + \frac{mg}{k}\right) - \frac{mg^2}{k}$$

$$v = \sqrt{2gL + \frac{mg^2}{k}}$$

0.2

0.5

(c) Time to come to rest = time in free fall + time in SHM of rope to stop stretching 0.1

$$\text{Length of free fall} = L = \tfrac{1}{2}gt_{\text{f}}^2$$

$$\text{Therefore } t_{\text{f}} = \sqrt{\frac{2L}{g}}$$

0.2

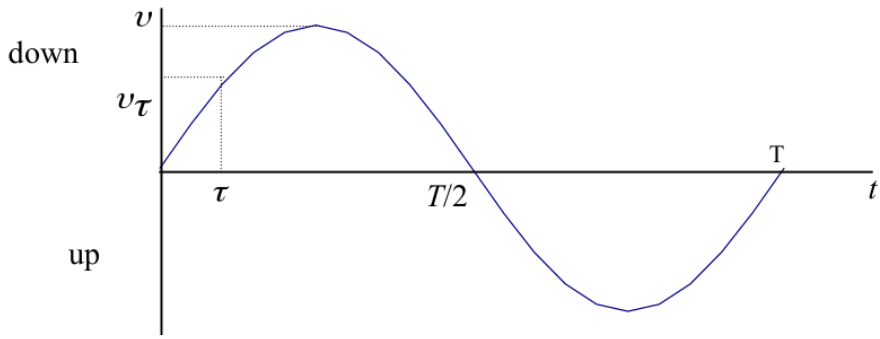
The jumper enters the SHM with free fall velocity $= gt_{\mathrm{f}} = \sqrt{2gL} = v_\tau$

0.1

$$\text{Period of SHM} = 2\pi\sqrt{\frac{m}{k}} = T$$

0.1

We represent a full SHM cycle by



The jumper enters the SHM at time τ given by

$$\tau = \frac{1}{\omega}\sin^{-1}\frac{v_\tau}{v} = \frac{1}{\omega}\sin^{-1}\frac{\sqrt{2gL}}{v}$$

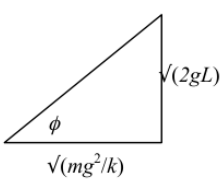
0.2

Jumper comes to rest at one half cycle of the SHM at total time given by

$$= t_{\mathrm{f}} + (T/2 - \tau)$$

0.1

$$= \sqrt{\frac{2L}{g}} + \pi\sqrt{\frac{m}{k}} - \frac{1}{\omega}\sin^{-1}\frac{\sqrt{2gL}}{v}$$



$$= \sqrt{\frac{2L}{g}} + \pi\sqrt{\frac{m}{k}} - \frac{1}{\omega}\sin^{-1}\frac{\sqrt{2gL}}{\sqrt{2gL + mg^2/k}}$$

$$= \sqrt{\frac{2L}{g}} + \sqrt{\frac{m}{k}}\left\{\pi - \sin^{-1}\frac{\sqrt{2gL}}{\sqrt{2gL + mg^2/k}}\right\}$$

This is the same as

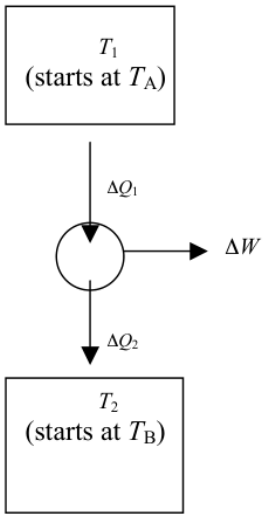
$$= \sqrt{\frac{2L}{g}} + \sqrt{\frac{m}{k}}\left\{\frac{\pi}{2} + \cos^{-1}\frac{\sqrt{2gL}}{\sqrt{2gL + mg^2/k}}\right\}$$

$$= \sqrt{\frac{2L}{g}} + \sqrt{\frac{m}{k}}\tan^{-1}\left\{-\sqrt{\frac{2kL}{mg}}\right\}$$

0.2

1.0

B Heat Engine Question



In calculating work obtainable, we assume no loss (friction etc.) in engine working.
ΔQ_1 = energy from body A
$= -ms\Delta T_1$ (ΔT_1 –ve)
$\Delta Q_2 = ms\Delta T_2$ (ΔT_2 +ve)

(a) For maximum amount of mechanical energy assume Carnot engine

$$\frac{\Delta Q_1}{T_1} = \frac{\Delta Q_2}{T_2} \quad \text{throughout operation (second law)}$$

0.2

But $\Delta Q_1 = -ms\Delta T_1$ and $\Delta Q_2 = ms\Delta T_2$ 0.2

$$-ms\int_{T_A}^{T_0}\frac{dT_1}{T_1} = ms\int_{T_B}^{T_0}\frac{dT_2}{T_2}$$

0.1

$$\ln\frac{T_A}{T_0} = \ln\frac{T_0}{T_B}$$

0.1

$$T_0^2 = T_A T_B$$

$$T_0 = \sqrt{T_A T_B}$$

0.2

0.8

$$Q_1 = -ms\int_{T_A}^{T_0} \mathrm{d}T_1 = ms(T_A - T_0)$$

0.2

$$Q_2 = ms\int_{T_B}^{T_0} \mathrm{d}T_2 = ms(T_0 - T_B)$$

0.1

$$W = Q_1 - Q_2$$

0.2

$$W = ms(T_A - T_0 - T_0 + T_B) = ms(T_A + T_B - 2T_0) = ms\left(T_A + T_B - 2\sqrt{T_A T_B}\right)$$

$$\text{or} \quad ms\left(\sqrt{T_\text{A}} - \sqrt{T_\text{B}}\right)^2$$

0.2

0.7

(d) Numerical example:

Mass = volume × density

$$\text{W} = 2.50 \times 1.00 \times 10^3 \times 4.19 \times 10^3 \times \left(350 + 300 - 2\sqrt{350 \times 300}\right) \text{ J}$$

$$= 20 \times 10^6 \text{ J}$$

$$= 20 \text{ MJ}$$

0.5

0.5

C Radioactivity and age of the Earth

(a) $N = N_0 \text{e}^{-\lambda}$ N_0 = original number 0.1

$$n = N_0(1 - \text{e}^{-\lambda t})$$

0.1

$$\text{Therefore } n = N\text{e}^{\lambda\text{t}}(1 - \text{e}^{-\lambda\text{t}}) = N(\text{e}^{\lambda\text{t}} - 1)$$

0.1

So $n = N(2^{t/\tau} - 1)$ where τ is half-life

$$\text{or as } \lambda = \frac{\ln 2}{T} = \frac{0.6931}{T}, \ \text{n} = \text{N}\left(e^{\frac{0.6931t}{T}} - 1\right)$$

0.1

$^{206}n = {}^{238}N(2^{t/4.50} - 1)$ or $^{206}n = {}^{238}N(e^{0.1540t} - 1)$ where time t is in 10^9 years 0.1

0.5

(b) $^{207}n = {}^{235}N(2^{t/0.710} - 1)$ or $^{207}n = {}^{235}N(e^{0.9762t} - 1)$ 0.1

0.1

(c) In mixed uranium (i.e. containing Pb of both natural and radioactive origin)

204 : 206 : 207 have proportions 1.00 : 29.6 : 22.6
In pure lead (no radioactivity) 1.00 : 17.9 : 15.5

Therefore for radioactively produced lead by subtraction

206 : 207 11.7 : 7.1 0.2

Dividing equations from (a) and (b) gives

$$\frac{^{206}n}{^{207}n} = \frac{^{238}N}{^{235}N}\left\{\frac{2^{t/4.50} - 1}{2^{t/0.710} - 1}\right\} \text{ or } \frac{^{206}n}{^{207}n} = \frac{^{238}N}{^{235}N}\left\{\frac{e^{0.1540t} - 1}{e^{0.9762t} - 1}\right\}$$

0.1

$$\frac{11.7}{7.1} = 137\left\{\frac{2^{T/4.50}-1}{2^{T/0.710}-1}\right\} \text{ or } \frac{11.7}{7.1} = 137\left\{\frac{e^{0.1540T}-1}{e^{0.9762T}-1}\right\}$$

0.1

$$0.0120\left\{2^{T/0.710}-1\right\} = \left\{2^{T/4.50}-1\right\} \text{ or } 0.0120\left\{\mathrm{e}^{0.9762T}-1\right\} = \left\{\mathrm{e}^{0.1540T}-1\right\}$$

0.1

0.5

(d) Assume $T >> 4.50 \times 10^9$ and ignore 1 in both brackets: 0.2

$$0.0120\left\{2^{T/0.710}\right\} = \left\{2^{T/4.50}\right\} \text{ or } 0.0120\left\{\mathrm{e}^{0.9762T}\right\} = \left\{\mathrm{e}^{0.1540T}\right\}$$

$$0.0120 = \left\{2^{T/4.50-T/0.710}\right\} = 2^{T(0.222-1.4084)} = 2^{-1.1862T}$$

$$T = -\frac{\log 0.0120}{\log 2 \times 1.1862} = 5.38$$

$T = 5.38 \times 10^9$ years

or $0.0120 = \mathrm{e}^{-0.8222\mathrm{T}}$ $\quad \mathrm{T} = \frac{\ln 0.0120}{-0.8222} = \frac{-4.4228}{-0.8222} = 5.38$

$T = 5.38 \times 10^9$ years 0.2

0.4

(e) T is not $>> 4.50 \times 10^9$ years but is $> 0.71 \times 10^9$ years 0.1

We can insert the approximate value for T (call it $T* = 5.38 \times 10^9$ years) in the $2^{T/4.50}$ term and obtain a better value by iteration in the rapidly changing $2^{T/0.710}$ term). We now leave in the –1's, although the –1 on the right-hand side has little effect and may be omitted). 0.1

$$\text{Either} \quad 0.0120\left(\left(2^{T/0.710}-1\right) = 2^{T*/4.50}-1\right.$$

$$2^{T/0.710}-1 = \frac{2^{1.1956}-1}{0.0120} = \frac{2.2904-1}{0.0120} = 107.5$$

$$T = 0.710\,\frac{\log 108.5}{\log 2} = 4.80(0)$$

0.2

Put T* = 4.80(0) × 10^9 years

$$2^{T/0.710} = \frac{2^{1.0668}-1}{0.0120} = \frac{2.0948-1}{0.0120} = 91.2$$

$$T = 0.710\,\frac{\log 91.2}{\log 2} = 4.62(3)$$

Further iteration gives 4.52 ***0.1***

or

$$0.0120(\mathrm{e}^{0.9762T} - 1) = (\mathrm{e}^{0.1540T^*} - 1) \quad \text{and similar}$$

So more accurate answer for* T *to be in range 4.6 × 10⁹ years to 4.5 × 10⁹ years (either acceptable).

0.5

D Spherical charge

(a) Charge density $= \rho = \dfrac{Q}{\frac{4}{3}\pi R^3}$ within sphere 0.3

$x \leq R$ Field at distance x:

$$E = \frac{\frac{4}{3}\pi x^3 \rho}{4\pi\varepsilon_0 x^2} = \frac{Qx}{4\pi\varepsilon_0 R^3}$$

0.3

$x > R$ Field at distance x from the centre: $E = \dfrac{Q}{4\pi\varepsilon_0 x^2}$ 0.2

0.8

(b) *Method 1*

Energy density is $\frac{1}{2}\varepsilon_0 E^2$. 0.1

$x \leq R$

Energy in a thin shell of thickness δx at radius x is given by

$$= \frac{1}{2}\varepsilon_0 E^2\, 4\pi x^2\, \delta x = \frac{1}{2}\, 4\pi\varepsilon_0\, \frac{Q^2 x^2}{(4\pi\varepsilon_0)^2 R^6}\, x^2\, \delta x$$

0.1

Energy within the spherical volume $= \dfrac{1}{2}\dfrac{Q^2}{(4\pi\varepsilon_0)R^6}\displaystyle\int_{x=0}^{x=R} x^4\, \mathrm{d}x = \frac{1}{40}\frac{Q^2}{\pi\varepsilon_0}\frac{1}{R}$ 0.2

$x > R$

Energy within spherical shell $= \dfrac{1}{2}\varepsilon_0 E^2\, 4\pi x^2\, \delta x = \dfrac{1}{2}\, 4\pi\varepsilon_0\, \dfrac{Q^2}{(4\pi\varepsilon_0)^2 x^4}\, x^2\, \delta x$ 0.1

Energy within the spherical volume for $x > R$

$$= \frac{1}{2}\frac{Q^2}{(4\pi\varepsilon_0)}\int_{x=R}^{x=\infty} \frac{1}{x^2}\, \mathrm{d}x = \frac{1}{8}\frac{Q^2}{\pi\varepsilon_0}\frac{1}{R}$$

0.2

$$\text{Total energy associated with the charge distribution} = \frac{1}{40}\frac{Q^2}{\pi\varepsilon_0}\frac{1}{R}$$

$$+\frac{1}{8}\frac{Q^2}{\pi\varepsilon_0}\frac{1}{R}$$

$$= \frac{3}{20}\frac{Q^2}{\pi\varepsilon_0}\frac{1}{R}$$

0.1

0.8

Method 2

A shell with charge $4\pi x^2\,\delta x\,\rho$ moves from ∞ to the surface of a sphere radius x where the electric potential is 0.1

$$\frac{\frac{4}{3}\pi x^3\rho}{4\pi\varepsilon_0 x} = \frac{x^2\rho}{3\varepsilon_0}$$

0.2

and will therefore gain electrical potential energy $\left(\frac{x^2\rho}{3\varepsilon_0}\right)\left(4\pi r^2\rho\right)\delta x$ 0.1

$$\text{Total energy of complete sphere} = \int_{x=0}^{x=R} \frac{4\pi\rho^2x^4}{3\varepsilon_0}\,dx = \frac{4}{15}\frac{\pi\rho^2R^5}{\varepsilon_0}$$

0.2

Putting Q = charge on sphere $= \frac{4}{3}\pi R^3\rho$, $\rho = \frac{3Q}{4\pi R^3}$

$$\text{So that total energy is} = \frac{4}{15}\pi\left(\frac{9Q^2}{16\pi^2R^6}\right)\frac{R^5}{\varepsilon} = \frac{3}{20}\frac{Q^2}{\pi\varepsilon_0 R}$$

0.2

0.8

(c) Binding energy $E_{\text{binding}} = E_{\text{electric}} - E_{\text{nuclear}}$ 0.1

Binding energy is a negative energy

Therefore $-8.768 = E_{\text{electric}} - 10.980$ MeV per nucleon

$E_{\text{electric}} = 2.212$ MeV per nucleon 0.1

$$\text{Radius of cobalt nucleus is given by } R = \frac{3}{20}\frac{Q^2}{\pi\varepsilon_0 E^{total}_{electric}}$$

$$= \frac{3\times 27^2\times(1.60\times10^{-19})^2}{20\times\pi\times8.85\times10^{-12}\times2.212\times10^6\times57\times1.60\times10^{-19}}\ \text{m}$$

$$= 5.0\times10^{-15}\ \text{m}$$

0.2

0.4

E E.M. Induction

Method 1 Equating energy

Horizontal component of magnetic field B inducing emf in ring:

$B = 44.5\times10^{-6}\cos 64°$ 0.2

Magnetic flux through ring at angle $\theta = B\pi a^2\sin\theta$

where a = radius of ring 0.1

Instantaneous emf $= \frac{\mathrm{d}\phi}{\mathrm{d}t} = B\pi a^2\frac{\mathrm{d}\sin\omega\mathrm{t}}{\mathrm{d}t}$ where ω = angular velocity

$$= B\pi a^2\omega\cos\omega t = B\pi a^2\omega\cos\theta$$

0.1

R.m.s. emf over 1 revolution $= \dfrac{B\pi a^2\omega}{\sqrt{2}}$ ***0.2***

Average resistive heating of ring $= \dfrac{B^2\pi^2a^4\omega^2}{2R}$ 0.1

Moment of inertia $= \frac{1}{2}ma^2$ 0.1

Rotational energy $= \frac{1}{4}ma^2\omega^2$ where m = mass of ring 0.1

$$\textbf{\textit{Power producing change in }}\omega = \frac{\mathrm{d}}{\mathrm{d}t}\left\{\frac{1}{4}ma^2\omega^2\right\} = \frac{1}{4}ma^2\,2\omega\,\frac{\mathrm{d}\omega}{\mathrm{d}t}$$

0.1

$$\text{Equating:}\quad \frac{1}{2}ma^2\omega\frac{\mathrm{d}\omega}{\mathrm{d}t} = -\frac{B^2\pi^2a^4\omega^2}{2R}$$

0.1

$$\frac{\mathrm{d}\omega}{\omega} = -\frac{B^2\pi^2a^2}{mR}\,\mathrm{d}t$$

0.1

If T is time for angular velocity to halve,

$$\int_{\omega}^{\omega/2}\frac{\mathrm{d}\omega}{\omega} = -\int_0^T\frac{B^2\pi^2a^2}{mR}\,\mathrm{d}t$$

0.1

$$\ln 2 = \frac{B^2\pi^2a^2}{mR}\,T$$

0.2

But R $= \dfrac{2\pi a\rho}{\mathrm{A}}$ where A is cross-sectional area of copper ring 0.1

$m = 2\pi adA$ (d = density) 0.1

$$\ln 2 = \frac{B^2\pi^2a^2T}{\dfrac{2\pi a\rho}{A}2\pi adA} = \frac{B^2T}{4\rho d}$$

0.1

$$T = \frac{4\rho d\ln 2}{B^2} = \frac{4\times 1.70\times 10^{-8}\times 8.90\times 10^3\times 0.6931}{(44.5\times 10^{-6}\times 0.4384)^2}\ \mathrm{s}$$

$$= 1.10(2)\times 10^6\ \mathrm{s}\ (= 306\ \mathrm{hr}\ = 12\ \mathrm{days}\ 18\ \mathrm{hr})$$

0.2

2.0

(Part E)

Method 2 Back Torque

Horizontal component of magnetic field $= B = 44.5\times 10^{-6}\cos 64°$ 0.2

Cross-section of area of ring is A
Radius of ring $= a$
Density of ring $= d$

Resistivity $= \rho$
ω = angular velocity (ω positive when clockwise)

Resistance $R = \rho\dfrac{2\pi a}{A}$ 0.1

Mass of ring m $= 2\pi aAd$ 0.1

Moment of inertia $= M = \frac{1}{2}ma^2$ 0.1

Magnetic flux through ring at angle $\theta = B\pi a^2 \sin\theta$ 0.1

Instantaneous emf $= \dfrac{\mathrm{d}\phi}{\mathrm{d}t} = B\pi a^2 \dfrac{\mathrm{d}\sin\omega t}{\mathrm{d}t} = B\pi a^2\omega\cos\omega t = B\pi a^2\omega\cos\theta$ 0.1

Induced current $=$ I $= B\pi a^2\cos\theta/R$

Torque opposing motion $= (B\pi a^2\cos\theta)I = \dfrac{1}{R}\left(B\pi a^2\right)^2\omega\cos^2\theta$ 0.1

Work done in small $\delta\theta = \dfrac{1}{R}\left(B\pi a^2\right)^2\omega\,\frac{1}{2}(\cos 2\theta + 1)\,\delta\theta$ 0.1

Average torque $=$ (work done in 2π revolution)$/2\pi$

$$= \frac{1}{2\pi R}\left(B\pi\pi^2\right)^2\omega\,\frac{1}{2}2\pi = \frac{1}{2R}\left(B\pi\pi^2\right)^2\omega$$

0.1

This equals $M\dfrac{\mathrm{d}\omega}{\mathrm{d}t}$ so that

$$M\frac{\mathrm{d}\omega}{\mathrm{d}t} = -\frac{B(\pi a^2)B(\pi a^2)^{\frac{1}{2}}}{(\rho/A)(2\pi a)}\,\omega$$

0.2

$$\frac{1}{2}(2\pi aAd)a^2\frac{\mathrm{d}\omega}{\mathrm{d}t} = -\frac{B^2(\pi a^2)^2A}{4\rho\pi a}\,\omega$$

$$\frac{\mathrm{d}\omega}{\mathrm{d}t} = -\frac{B^2}{4\rho d}\,\omega$$

0.2

$$\int_{\omega}^{\omega/2}\frac{\mathrm{d}\omega}{\omega} = \int_0^T \frac{B^2}{4\rho d}\,\mathrm{d}t$$

0.2

$$\ln 2 = \frac{B^2T}{4\rho d}$$

0.2

$$T = \frac{4\rho d\ln 2}{B^2} = \frac{4\times 1.70\times 10^{-8}\times 8.90\times 10^3\times 0.6931}{(44.5\times 10^{-6}\times 0.4384)^2}\ \mathrm{s}$$

$$= 1.10(2)\times 10^6\,\mathrm{s}\ = 306\ \mathrm{hr}\ = 12\ \mathrm{days}\ 18\ \mathrm{hr}$$

0.2

2.0