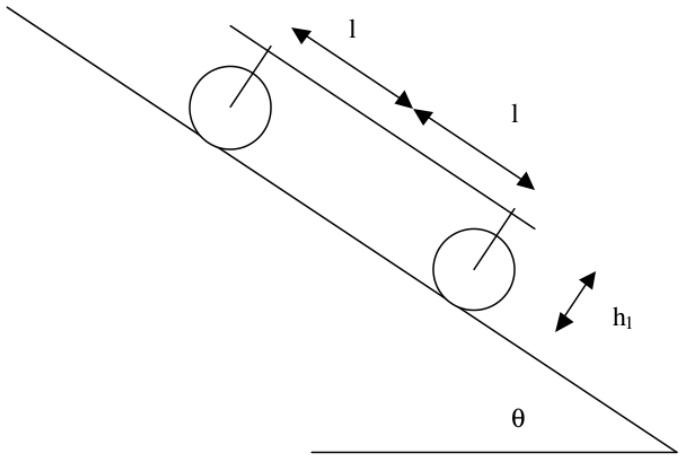
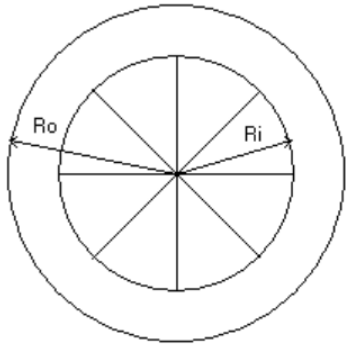
SOLUTION T3 : . A Heavy Vehicle Moving on An Inclined Road



To simplify the model we use the above figure with $h_1 = h + 0.5\,t$
$R_o = R$

1. Calculation of the moment inertia of the cylinder

$R_i = 0.8\,R_o$
Mass of cylinder part : $m_{cylinder} = 0.8\,M$
Mass of each rod : $m_{rod} = 0.025\,M$



$$I = \oint_{wholepart} r^2\,dm = \oint_{cyl.shell} r^2\,dm + \oint_{\text{rod1}} r^2\,dm + \cdots + \oint_{rod\,n} r^2\,dm$$

0.4 pts

$$\oint_{cyl.shell} r^2\,dm = 2\pi\sigma \int_{Ri}^{Ro} r^3\,dr = 0.5\pi\sigma\,(R_o^4 - R_i^4) = 0.5\,m_{cylinder}(R_o^2 + R_i^2)$$
$$= 0.5(0.8M)R^2(1 + 0.64) = 0.656MR^2$$

0.5 pts

$$\oint_{\text{rod}} r^2\,dm = \lambda \int_0^{Rin} r^2\,dr = \frac{1}{3}\lambda R_{in}^3 = \frac{1}{3} m_{rod} R_{in}^2 = \frac{1}{3}\,0.025M\,(0.64R^2) = 0.00533MR^2$$

0.5 pts

The moment inertia of each wheel becomes

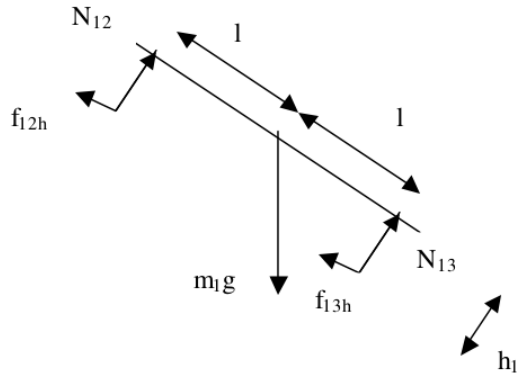
$$I = 0.656MR^2 + 8x0.00533MR^2 = 0.7MR^2$$

0.1 pts

2. Force diagram and balance equations:

To simplify the analysis we devide the system into three parts: frame (part1) which mainly can be treated as flat homogeneous plate, rear cylinders (two cylinders are treated collectively as part 2 of the system), and front cylinders (two front cylinders are treated collectively as part 3 of the system).

Part 1 : Frame



0.4 pts

The balance equation related to the forces work to this parts are:

Required conditions:

Balance of force in the horizontal axis

$$m_1 g \sin\theta - f_{12h} - f_{13h} = m_1 a \tag{1}$$

0.2 pts

Balance of force in the vertical axis

$$m_1 g \cos\theta = N_{12} + N_{13} \tag{2}$$

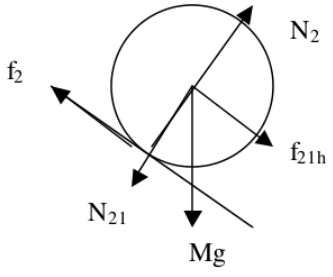
0.2 pts

Then torsi on against O is zero, so that

$$\mathrm{N}_{12} l - \mathrm{N}_{13} l + f_{12h} h_1 + f_{13h} h_1 = 0 \tag{3}$$

0.2 pts

Part two : Rear cylinder



0.25 pts

From balance condition in rear wheel :

$$\mathrm{f}_{21\mathrm{h}} - f_2 + Mg \sin\theta = Ma \tag{4}$$

0.15 pts

$$N_2 - N_{21} - Mg\cos\theta = 0 \tag{5}$$

0.15 pts

For pure rolling:

$$f_2 R = I\alpha_2 = I\frac{a_2}{R}$$

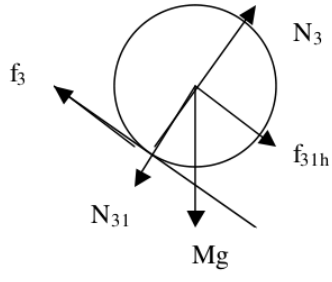
$$\text{or} \quad f_2 = \frac{I}{R^2}\,a \tag{6}$$

For rolling with sliding:

$$F_2 = u_k N_2 \tag{7}$$

0.2 pts

Part Three : Front Cylinder:



0.25 pts

From balance condition in the front whee l :

$$f_{31h} - f_3 + Mg\sin\theta = Ma \tag{8}$$

0.15 pts

$$N_3 - N_{31} - Mg\cos\theta = 0 \tag{9}$$

0.15 pts

For pure rolling:

$$f_3 R = I\alpha_3 = I\frac{a_3}{R}$$

$$\text{or} \quad f_3 = \frac{I}{R^2}\,a \tag{10}$$

For rolling with sliding:

$$F_3 = u_k N_3 \tag{11}$$

0.2 pts

3. From equation (2), (5) and (9) we get

$$m_1 g\cos\theta = N_2 - m_2 g\cos\theta + N_3 - m_3 g\cos\theta$$

$$N_2 + N_3 = (m_1 + m_2 + m_3)g\cos\theta = 7Mg\cos\theta \tag{12}$$

And from equation (3), (5) and (8) we get

$$(N_3 - Mg\cos\theta)\,l - (N_2 - Mg\cos\theta)\,l = h_1(f_2 + Ma - Mg\sin\theta + f_3 + Ma - Mg\sin\theta)$$

$$(N_3 - N_2) = h_1(f_2 + 2Ma - 2Mg\sin\theta + f_3)/l$$

Equations 12 and 13 are given **0.25 pts**

CASE ALL CYLINDER IN PURE ROLLING

From equation (4) and (6) we get

$$f_{21h} = (I/R^2)a + Ma - Mg\sin\theta \tag{14}$$

0.2 pts

From equation (8) and (10) we get

$$f_{31h} = (I/R^2)a + Ma - Mg\sin\theta \tag{15}$$

0.2 pts

Then from eq. (1) , (14) and (15) we get

$$5Mg\sin\theta - \{(I/R^2)a + Ma - Mg\sin\theta\} - \{(I/R^2)a + Ma - Mg\sin\theta\} = m_1 a$$

$$7Mg\sin\theta = (2I/R^2 + 7M)a$$

$$a = \frac{7Mg\sin\theta}{7M + 2\frac{I}{R^2}} = \frac{7Mg\sin\theta}{7M + 2\frac{0.7MR^2}{R^2}} = 0.833g\sin\theta \tag{16}$$

0.35 pts

$$N_3 = \frac{7M}{2}g\cos\theta + \frac{h_1}{l}\left[\left(M + \frac{I}{R^2}\right) \times 0.833g\sin\theta - Mg\sin\theta\right]$$

$$= 3.5Mg\cos\theta + \frac{h_1}{l}\left[(M + 0.7M) \times 0.833g\sin\theta - Mg\sin\theta\right]$$

$$= 3.5Mg\cos\theta + 0.41\,\frac{h_1}{l}\,Mg\sin\theta$$

$$N_2 = \frac{7M}{2}g\cos\theta - \frac{h_1}{l}\left[\left(\frac{I}{R^2} + M\right) \times 0.833g\sin\theta - Mg\sin\theta\right]$$

$$= 3.5g\cos\theta - \frac{h_1}{l}\left[(0.7M + M)\frac{7Mg\sin\theta}{0.7M + 7M} - 2Mg\sin\theta\right]$$

$$= 3.5g\cos\theta - 0.41\,\frac{h_1}{l}\,Mg\sin\theta$$

0.2 pts

The Conditions for pure rolling:

$$f_2 \le \mu_s N_2 \quad \text{and} \quad f_3 \le \mu_s N_3$$

$$\frac{I_2}{R_2^2}\,a \le \mu_s N_2 \quad \text{and} \quad \frac{I_3}{R_3^2}\,a \le \mu_s N_3$$

0.2 pts

The left equation becomes

$$0.7M \times 0.833g\sin\theta \le \mu_s\left(3.5Mg\cos\theta - 0.41\,\frac{h_1}{l}\,Mg\sin\theta\right)$$

$$\tan\theta \leq \frac{3.5\mu_s}{0.5831 + 0.41\mu_s \frac{h_1}{l}}$$

While the right equation becomes

$$0.7m \times 0.833g \sin\theta \leq \mu_s \left(3.5\text{mg}\cos\theta + 0.41\,\frac{h_1}{l}\,\text{mg}\sin\theta\right)$$

$$\tan\theta \leq \frac{3.5\mu_s}{0.5831 - 0.41\mu_s \frac{h_1}{l}} \tag{17}$$

0.1 pts

CASE ALL CYLINDER SLIDING

From eq. (4)

$$\text{f}_{21\text{h}} = \text{Ma} + \text{u}_\text{k}\text{N}_2 - \text{Mg}\sin\theta \tag{18}$$

0.15 pts

From eq. (8)

$$\text{f}_{31\text{h}} = \text{Ma} + \text{u}_\text{k}\text{N}_3 - \text{Mg}\sin\theta \tag{19}$$

0.15 pts

From eq. (18) and 19 :

$$5\text{Mg}\sin\theta - (\text{Ma} + \text{u}_\text{k}\text{N}_2 - \text{Mg}\sin\theta) - (\text{Ma} + \text{u}_\text{k}\text{N}_3 - \text{Mg}\sin\theta) = \text{m}_1\text{a}$$

$$a = \frac{7Mg\sin\theta - \mu_k N_2 - \mu_k N_3}{7M} = g\sin\theta - \frac{\mu_k(N_2 + N_3)}{7M} \tag{20}$$

0.2 pts

$$N_3 + N_2 = 7Mg\cos\theta$$

From the above two equations we get :

$$a = g\sin\theta - \mu_k g\cos\theta$$

0.25 pts

The Conditions for complete sliding: are the opposite of that of pure rolling

$$\begin{array}{rcl} f_2 \rangle \mu_s N_2' & \text{and} & f_3 \rangle \mu_s N_3' \\ \frac{\text{I}_2}{\text{R}_2^2}\,\text{a} \rangle \mu_s N_2' & \text{and} & \frac{\text{I}_3}{\text{R}_3^2}\,\text{a} \rangle \mu_s N_3' \end{array} \tag{21}$$

0.2 pts

Where N_2' and N_3' is calculated in case all cylinder in pure rolling. 0.1 pts

Finally weget

$$\tan\theta \rangle \frac{3.5\mu_s}{0.5831 + 0.41\mu_s \frac{h_1}{l}} \qquad \text{and} \qquad \tan\theta \rangle \frac{3.5\mu_s}{0.5831 - 0.41\mu_s \frac{h_1}{l}}$$

0.2 pts

The left inequality finally become decisive.

CASE ONE CYLINDER IN PURE ROLLING AND ANOTHER IN SLIDING CONDITION

{ For example R_3 (front cylinders) pure rolling while R_2 (Rear cylinders) sliding}

From equation (4) we get

$$F_{21h} = m_2 a + u_k N_2 - m_2 g \sin\theta \tag{22}$$

0.15 pts

From equation (5) we get

$$f_{31h} = m_3 a + (I/R^2)a - m_3 g \sin\theta \tag{23}$$

0.15 pts

Then from eq. (1) , (22) and (23) we get

$$m_1 g \sin\theta - \{m_2 a + u_k N_2 - m_2 g \sin\theta\} - \{m_3 a + (I/R^2)a - m_3 g \sin\theta\} = m_1 a$$

$$m_1 g \sin\theta + m_2 g \sin\theta + m_3 \sin\theta - u_k N_2 = (I/R^2 + m_3)a + m_2 a + m_1 a$$

$$5Mg \sin\theta + Mg \sin\theta + Mg \sin\theta - u_k N_2 = (0.7M + M)a + Ma + 5Ma$$

$$a = \frac{7Mg \sin\theta - \mu_k N_2}{7.7M} = 0.9091 g \sin\theta - \frac{\mu_k N_2}{7.7M} \tag{24}$$

0.2 pts

$$N_3 - N_2 = \frac{h_1}{l}\left(\mu_k N_2 + \frac{I}{R^2}a + 2Ma - 2Mg \sin\theta\right)$$

$$N_3 - N_2 = \frac{h_1}{l}\left(\mu_k N_2 + 2.7M \times 0.9091 g \sin\theta - 2.7\mu_k N_2/7.7 - 2Mg \sin\theta\right)$$

$$N_3 - N_2\left(1 + 0.65\mu_k \frac{h_1}{l}\right) = 0.4546 Mg \sin\theta$$

$$N_3 + N_2 = 7Mg \cos\theta$$

Therefore we get

$$\begin{aligned} N_2 &= \frac{7Mg \cos\theta - 0.4546Mg \sin\theta}{2 + 0.65\mu_k \frac{h_1}{l}} \\ N_3 &= 7Mg \cos\theta - \frac{7Mg \cos\theta - 0.4546Mg \sin\theta}{2 + 0.65\mu_k \frac{h_1}{l}} \end{aligned} \tag{25}$$

0.3 pts

Then we can substitute the results above into equation (16) to get the following result

$$a = 0.9091 g \sin\theta - \frac{\mu_k N_2}{7.7M} = 0.9091 g \sin\theta - \frac{\mu_k}{7.7}\frac{7g \cos\theta - 0.4546 g \sin\theta}{2 + 0.65\mu_k \frac{h_1}{l}} \tag{26}$$

0.2 pts

The Conditions for this partial sliding is:

$$\begin{aligned} f_2 \le \mu_s N_2' \quad &\text{and} \quad f_3 \rangle \mu_s N_3' \\ \frac{I}{R^2} a \le \mu_s N_2' \quad &\text{and} \quad \frac{I}{R^2} a \rangle \mu_s N_3' \end{aligned} \tag{27}$$

0.25 pts

where N_2' and N_3' are normal forces for pure rolling condition

4. Assumed that after rolling d meter all cylinder start to sliding until reaching the end of incline road (total distant is s meter). Assummed that ηmeter is reached in t_1 second.

$$v_{t1} = v_o + at_1 = 0 + a_1 t_1 = a_1 t_1$$
$$d = v_o t_1 + \tfrac{1}{2} a_1 t_1^2 = \tfrac{1}{2} a_1 t_1^2$$
$$t_1 = \sqrt{\frac{2d}{a_1}}$$

0.5 pts

$$v_{t1} = a_1 \sqrt{\frac{2d}{a_1}} = \sqrt{2da_1} = \sqrt{2d\, 0.833g \sin\theta} = \sqrt{1.666dg \sin\theta} \tag{28}$$

The angular velocity after rolling d meters is same for front and rear cylinders:

$$\omega_{t1} = \frac{v_{t1}}{R} = \frac{1}{R}\sqrt{1.666dg \sin\theta} \tag{29}$$

0.5 pts

Then the vehicle sliding untill the end of declining road. Assumed that the time needed by vehicle to move from d position to the end of the declining road is t_2 second.

$$v_{t2} = v_{t1} + a_2 t_2 = \sqrt{1.666dg \sin\theta} + a_2 t_2$$
$$s - d = v_{t1} t_2 + \tfrac{1}{2} a_2 t_2^2$$
$$t_2 = \frac{-v_{t1} + \sqrt{v_{t1}^2 + 2a_2(s-d)}}{a_2} \tag{30}$$

0.4 pts

$$v_{t2} = \sqrt{1.666dg \sin\theta} - v_{t1} + \sqrt{v_{t1}^2 + 2a_2(s-d)}$$

Inserting v_{t1} and a_2 from the previous results we get the final results.

For the angular velocity, while sliding they receive torsion:

$$\tau = \mu_k NR$$
$$\alpha = \frac{\tau}{I} = \frac{\mu_k NR}{I} \tag{31}$$
$$\omega_{t2} = \omega_{t1} + \alpha t_2 = \frac{1}{R}\sqrt{1.666dg \sin\theta} + \frac{\mu_k NR}{I}\frac{-v_{t1} + \sqrt{v_{t1}^2 + 2a_2(s-d)}}{a_2}$$

0.6 pts