Theoretical Question 3: ***Scanning Probe Microscope***

1. Answers

(a) $A = \dfrac{F_0}{\sqrt{m^2(\omega_0^2-\omega^2)^2+b^2\omega^2}}$ and $\tan\phi = \dfrac{b\omega_0}{m(\omega_0^2-\omega^2)}$. At $\omega=\omega_0$, $A = \dfrac{F_0}{b\omega_0}$ and $\phi = \dfrac{\pi}{2}$.

(b) A non-vanishing dc component exists only when $\omega = \omega_i$. In this case the amplitude of the dc signal will be $\frac{1}{2}V_{i0}V_{R0}\cos\phi_i$.

(c) $\dfrac{c_1c_2}{2}\dfrac{V_{R0}^2}{b\omega_0}$ at the resonance frequency ω_0.

(d) $\Delta m = 1.7\times10^{-18}\,\text{kg}$.

(e) $\omega_0' = \omega_0\left(1-\dfrac{c_3}{m\omega_0^2}\right)^{1/2}$.

(f)

$$d_0 = \left(k_e\frac{qQ}{m\omega_0\Delta\omega_0}\right)^{1/3}$$
$$d_0 = 41\,\text{nm}.$$

2. Solutions

(a) [1.5 points]

Substituting $z(t) = A\sin(\omega t-\phi)$ in the equation $m\dfrac{d^2z}{dt^2}+b\dfrac{dz}{dt}+m\omega_0^2 z = F_0\sin\omega t$ yields,

$$-m\omega^2\sin(\omega t-\phi)+b\omega\cos(\omega t-\phi)+m\omega_0^2\sin(\omega t-\phi) = \frac{F_0}{A}\sin\omega t\,. \tag{a1}$$

Collecting terms proportional to $\sin\omega t$ and $\cos\omega t$, one obtains

$$\left\{m(\omega_0^2-\omega^2)\cos\phi+b\omega\sin\phi-\frac{F_0}{A}\right\}\sin\omega t+\left\{-m(\omega_0^2-\omega^2)\sin\phi+b\omega\cos\phi\right\}\cos\omega t = 0 \tag{a2}$$

Zeroing the each curly square bracket produces

$$\tan\phi = \frac{b\omega}{m(\omega_0^2-\omega^2)}, \tag{a3}$$

$$A = \frac{F_0}{\sqrt{m^2(\omega_0^2-\omega^2)^2+b^2\omega^2}}. \tag{a4}$$

At $\omega=\omega_0$,

$$A = \frac{F_0}{b\omega_0} \quad\text{and}\quad \phi = \frac{\pi}{2}. \tag{a5}$$

(b) [1 point]

The multiplied signal is

$$\begin{aligned} &V_{i0}\sin(\omega_i t-\phi_i)V_{R0}\sin(\omega t)\\ &= \frac{1}{2}V_{i0}V_{R0}\left[\cos\{(\omega_i-\omega)t-\phi_i\}-\cos\{(\omega_i+\omega)t-\phi_i\}\right]\end{aligned} \tag{b1}$$

A non-vanishing dc component exists only when $\omega=\omega_i$. In this case the amplitude of the dc signal will be

$$\frac{1}{2}V_{i0}V_{R0}\cos\phi_i\,. \tag{b2}$$

(c) [1.5 points]

Since the lock-in amplifier measures the ac signal of the same frequency with its reference signal, the frequency of the piezoelectric tube oscillation, the frequency of the cantilever, and the frequency of the photodiode detector should be same. The magnitude of the input signal at the resonance is

$$V_{i0} = c_2 \frac{F_0}{b\omega_0} = \frac{c_1 c_2 V_{R0}}{b\omega_0} \,. \tag{c1}$$

Then, since the phase of the input signal is $-\frac{\pi}{2} + \frac{\pi}{2} = 0$ at the resonance, $\phi_i = 0$ and the lock-in amplifier signal is

$$\frac{1}{2} V_{i0} V_{R0} \cos 0 = \frac{c_1 c_2}{2} \frac{V_{R0}^2}{b\omega_0} \,. \tag{c2}$$

(d) [2 points]

The original resonance frequency $\omega_0 = \sqrt{k/m}$ is shifted to

$$\sqrt{\frac{k}{m+\Delta m}} = \sqrt{\frac{k}{m}} \left(1 + \frac{\Delta m}{m}\right)^{-1/2} \cong \sqrt{\frac{k}{m}} \left(1 - \frac{1}{2}\frac{\Delta m}{m}\right) = \omega_0 \left(1 - \frac{1}{2}\frac{\Delta m}{m}\right). \tag{d1}$$

Thus

$$\Delta\omega_0 = -\frac{1}{2}\omega_0 \frac{\Delta m}{m}. \tag{d2}$$

Near the resonance, by substituting $\phi \to \frac{\pi}{2} + \Delta\phi$ and $\omega_0 \to \omega_0 + \Delta\omega_0$ in Eq. (a3), the change of the phase due to the small change of ω_0 (not the change of ω) is

$$\tan\left(\frac{\pi}{2} + \Delta\phi\right) = -\frac{1}{\tan\Delta\phi} = \frac{b}{2m\Delta\omega_0} \,. \tag{d3}$$

Therefore,

$$\Delta\phi \approx \tan\Delta\phi = -\frac{2m\Delta\omega_0}{b} \,. \tag{d4}$$

From Eqs. (d2) and (d4),

$$\Delta m = \frac{b}{\omega_0}\Delta\phi = \frac{10^3 \cdot 10^{-12}}{10^6} \frac{\pi}{1800} = \frac{\pi}{1.8} 10^{-18} = 1.7 \times 10^{-18}\,\text{kg}. \tag{d5}$$

(e) [1.5 points]

In the presence of interaction, the equation of motion near the new equilibrium position h_0 becomes

$$m\frac{d^2 z}{dt^2} + b\frac{dz}{dt} + m\omega_0^2 z - c_3 z = F_0 \sin\omega t \tag{e1}$$

where we used $f(h) \approx f(h_0) + c_3 z$ with $z = h - h_0$ being the displacement from the new equilibrium position h_0. Note that the constant term $f(h_0)$ is cancelled at the new equilibrium position.

Thus the original resonance frequency $\omega_0 = \sqrt{k/m}$ will be shifted to

$$\omega_0' = \sqrt{\frac{k - c_3}{m}} = \sqrt{\frac{m\omega_0^2 - c_3}{m}} = \omega_0 \sqrt{1 - \frac{c_3}{m\omega_0^2}} \,. \tag{e3}$$

Hence the resonance frequency shift is given by

$$\Delta\omega_0 = \omega_0 \left[\sqrt{1 - \frac{c_3}{m\omega_0^2}} - 1\right] \,. \tag{e4}$$

(f) [2.5 points]

The maximum shift occurs when the cantilever is on top of the charge, where the interacting force is given by

$$f(h) = k_e \frac{qQ}{h^2} \,. \tag{f1}$$

From this,

$$c_3 = \left.\frac{df}{dh}\right|_{h=d_0} = -2k_e \frac{qQ}{d_0^3} \,. \tag{f2}$$

Since $\Delta\omega_0 << \omega_0$, we can approximate Eq. (e4) as

$$\Delta\omega_0 \approx -\frac{c_3}{2m\omega_0} \,. \tag{f3}$$

From Eqs. (f2) and (f3), we have

$$\Delta\omega_0 = -\frac{1}{2m\omega_0}\left(-2k_e \frac{qQ}{d_0^3}\right) = k_e \frac{qQ}{m\omega_0 d_0^3} \,. \tag{f4}$$

Here $q = e = -1.6 \times 10^{-19}$ Coulomb and $Q = 6e = -9.6 \times 10^{-19}$ Coulomb. Using the values provided,

$$d_0 = \left(k_e \frac{qQ}{m\omega_0 \Delta\omega_0}\right)^{1/3} = 4.1 \times 10^{-8}\,\mathrm{m} = 41\,\mathrm{nm}. \tag{f5}$$

Thus the trapped electron is 41 nm from the cantilever.

3. Mark Distribution

No.	Total Pt.	Partial Pt.	Contents
(a)	1.5	0.7	Equations for A and ϕ (substitution and manipulation)
		0.4	Correct answers for A and ϕ
		0.4	A and ϕ at ω_0
(b)	1.0	0.4	Equation for the multiplied signal
		0.3	Condition for the non-vanishing dc output
		0.3	Correct answer for the dc output
(c)	1.5	0.6	Relation between V_i and V_R
		0.4	Condition for the maximum dc output
		0.5	Correct answer for the magnitude of dc output
(d)	2.0	0.5	Relation between Δm and $\Delta\omega_0$
		1.0	Relations between $\Delta\omega_0$ (or Δm) and $\Delta\phi$
		0.5	Correct answer (Partial credit of 0.2 for the wrong sign.)
(e)	1.5	1.0	Modification of the equation with $f(h)$ and use of a proper approximation for the equation
		0.5	Correct answer
(f)	2.5	0.5	Use of a correct formula of Coulomb force
		0.3	Evaluation of c_3
		0.6	Use of the result in (e) for either $\Delta\omega_0$ or $\omega_0'^2 - \omega_0^2$
		0.6	Expression for d_0
		0.5	Correct answer
Total	10		