In physics, whenever we have an equality relation, both sides of the equation should be of the same type i.e. they must have the same dimensions. For example you cannot have a situation where the quantity on the right-hand side of the equation represents a length and the quantity on the left-hand side represents a time interval. Using this fact, sometimes one can nearly deduce the form of a physical relation without solving the problem analytically. For example if we were asked to find the time it takes for an object to fall from a height of h under the influence of a constant gravitational acceleration g, we could argue that one only needs to build a quantity representing a time interval, using the quantities g and h and the only possible way of doing this is $T = a\,(h/g)^{1/2}$. Notice that this solution includes an as yet undetermined coefficient a which is *dimensionless* and thus cannot be determined, using this method. This coefficient can be a number such as 1, 1/2, $\sqrt{3}$, π, or any other real number. This method of deducing physical relations is called *dimensional analysis*. In dimensional analysis the dimensionless coefficients are not important and we do not need to write them. Fortunately in most physical problems these coefficients are of the order of 1 and eliminating them does not change the order of magnitude of the physical quantities. Therefore, by applying the dimensional analysis to the above problem, one obtains $T = (h/g)^{1/2}$**.**

Generally, the dimensions of a physical quantity are written in terms of the dimensions of four fundamental quantities: M (mass), L (length), T (time), and K (temperature). The dimensions of an arbitrary quantity, x is denoted by $[x]$. As an example, to express the dimensions of velocity v, kinetic energy E_k, and heat capacity C_V we write: $[v] = LT^{-1}$, $[E_k] = ML^2T^{-2}$, $[C_V] = ML^2T^{-2}K^{-1}$**.**

1 Fundamental Constants and Dimensional Analysis

1.1 Find the dimensions of *the fundamental constants,* i.e. the Planck's constant, h, the speed of light, c, the universal constant of gravitation, G, and the Boltzmann constant, k_B, in terms of the dimensions of length, mass, time, and temperature. 0.8

The Stefan-Boltzmann law states that the black body emissive power which is the total energy radiated per unit surface area of a black body in unit time is equal to $\sigma\theta^4$ where σ is the Stefan-Boltzmann's constant and θ is the absolute temperature of the black body.

1.2 Determine the dimensions of the Stefan-Boltzmann's constant in terms of the dimensions of length, mass, time, and temperature. 0.5

The Stefan-Boltzmann's constant is not a fundamental constant and one can write it in terms of fundamental constants i.e. one can write $\sigma = a\,h^\alpha c^\beta G^\gamma k_B^\delta$. In this relation a is a dimensionless parameter of the order of 1. As mentioned before, the exact value of a is not significant from our viewpoint, so we will set it equal to 1.

1.3 Find α, β, γ, and δ using dimensional analysis. 1.0

2 Physics of Black Holes

In this part of the problem, we would like to find out some properties of black holes using dimensional analysis. According to a certain theorem in physics known as the *no hair theorem*, all the characteristics of the black hole which we are considering in this problem depend only on the mass of the black hole. One characteristic of a black hole is the area of its *event horizon*. Roughly speaking, the event horizon is the boundary of the black hole. Inside this boundary, the gravity is so strong that even light cannot emerge from the region enclosed by the boundary.

We would like to find a relation between the mass of a black hole, m, and the area of its event horizon, A. This area depends on the mass of the black hole, the speed of light, and the universal constant of gravitation. As in 1.3 we shall write $A = G^\alpha c^\beta m^\gamma$.

2.1 Use dimensional analysis to find α, β, and γ. 0.8

From the result of 2.1 it becomes clear that the area of the event horizon of a black hole increases with its mass. From a classical point of view, nothing comes out of a black hole and therefore in all physical processes the area of the event horizon can only

increase. In analogy with the second law of thermodynamics, Bekenstein proposed to assign entropy, S, to a black hole, proportional to the area of its event horizon i.e. $S = \eta A$. This conjecture has been made more plausible using other arguments.

2.2 Use the thermodynamic definition of entropy $dS = dQ/\theta$ to find the dimensions of entropy.
dQ is the exchanged heat and θ is the absolute temperature of the system. 0.2

2.3 As in 1.3, express the dimensioned constant η as a function of the fundamental constants h, c, G, and k_B. 1.1

*Do **not** use dimensional analysis for the rest of problem, but you may use the results you have obtained in previous sections.*

3 Hawking Radiation

With a semi-quantum mechanical approach, Hawking argued that contrary to the classical point of view, black holes emit radiation similar to the radiation of a black body at a temperature which is called the *Hawking temperature*.

3.1 Use $E = mc^2$, which gives the energy of the black hole in terms of its mass, and the laws of thermodynamics to express the Hawking temperature θ_H of a black hole in terms of its mass and the fundamental constants. Assume that the black hole does no work on its surroundings. 0.8

3.2 The mass of an isolated black hole will thus change because of the Hawking radiation. Use Stefan-Boltzmann's law to find the dependence of this rate of change on the Hawking temperature of the black hole, θ_H and express it in terms of mass of the black hole and the fundamental constants. 0.7

3.3 Find the time t^*, that it takes an isolated black hole of mass m to evaporate completely i.e. to lose all its mass. 1.1

From the viewpoint of thermodynamics, black holes exhibit certain exotic behaviors. For example the heat capacity of a black hole is negative.

3.4 Find the heat capacity of a black hole of mass m. 0.6

4 Black Holes and the Cosmic Background Radiation

Consider a black hole exposed to the cosmic background radiation. The cosmic background radiation is a black body radiation with a temperature θ_B which fills the entire universe. An object with a total area A will thus receive an energy equal to $\sigma\theta_B^4 \times A$ per unit time. A black hole, therefore, loses energy through Hawking radiation and gains energy from the cosmic background radiation.

4.1 Find the rate of change of a black hole's mass, in terms of the mass of the black hole, the temperature of the cosmic background radiation, and the fundamental constants. 0.8

4.2 At a certain mass, m^*, this rate of change will vanish. Find m^* and express it in terms of θ_B and the fundamental constants. 0.4

4.3 Use your answer to 4.2 to substitute for θ_B in your answer to part 4.1 and express the rate of change of the mass of a black hole in terms of m, m^*, and the fundamental constants. 0.2

4.4 Find the Hawking temperature of a black hole at thermal equilibrium with cosmic background radiation. 0.4

4.5 Is the equilibrium stable or unstable? Why? (Express your answer mathematically) 0.6