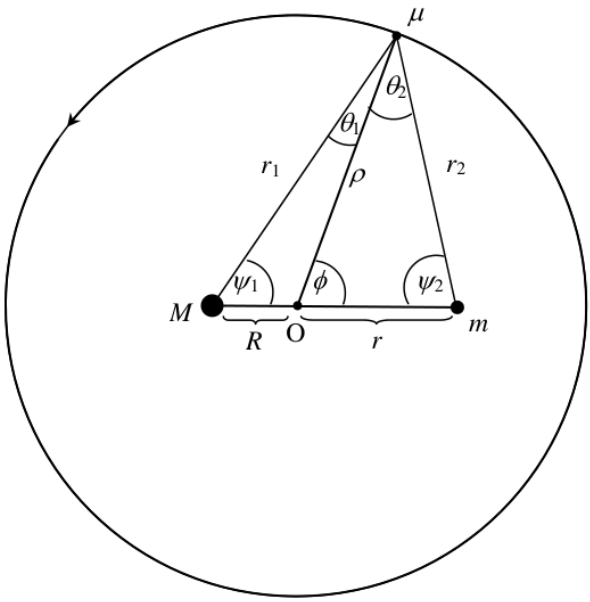
I. Solution



1.1 Let O be their centre of mass. Hence

$$MR - mr = 0 \tag{1}$$

$$\begin{aligned} m\omega_0^2 r &= \frac{GMm}{(R+r)^2} \\ M\omega_0^2 R &= \frac{GMm}{(R+r)^2} \end{aligned} \tag{2}$$

From Eq. (2), or using reduced mass,

$$\omega_0^2 = \frac{G(M+m)}{(R+r)^3}$$

$$\text{Hence, } \omega_0^2 = \frac{G(M+m)}{(R+r)^3} = \frac{GM}{r(R+r)^2} = \frac{Gm}{R(R+r)^2}. \tag{3}$$

1.2 Since μ is infinitesimal, it has no gravitational influences on the motion of neither M nor m . For μ to remain stationary relative to both M and m we must have:

$$\frac{GM\mu}{r_1^2}\cos\theta_1 + \frac{Gm\mu}{r_2^2}\cos\theta_2 = \mu\omega_0^2\rho = \frac{G(M+m)\mu}{(R+r)^3}\rho \tag{4}$$

$$\frac{GM\mu}{r_1^2}\sin\theta_1 = \frac{Gm\mu}{r_2^2}\sin\theta_2 \tag{5}$$

Substituting $\frac{GM}{r_1^2}$ from Eq. (5) into Eq. (4), and using the identity $\sin\theta_1\cos\theta_2 + \cos\theta_1\sin\theta_2 = \sin(\theta_1+\theta_2)$, we get

$$m\frac{\sin(\theta_1+\theta_2)}{r_2^2} = \frac{(M+m)}{(R+r)^3}\rho\sin\theta_1 \tag{6}$$

The distances r_2 and ρ , the angles θ_1 and θ_2 are related by two Sine Rule equations

$$\begin{aligned} \frac{\sin\psi_1}{\rho} &= \frac{\sin\theta_1}{R} \\ \frac{\sin\psi_1}{r_2} &= \frac{\sin(\theta_1+\theta_2)}{R+r} \end{aligned} \tag{7}$$

Substitute (7) into (6)

$$\frac{1}{r_2^3} = \frac{R}{(R+r)^4}\frac{(M+m)}{m} \tag{10}$$

Since $\frac{m}{M+m} = \frac{R}{R+r}$,Eq. (10) gives

$$r_2 = R + r \tag{11}$$

By substituting $\frac{Gm}{r_2^2}$ from Eq. (5) into Eq. (4), and repeat a similar procedure, we get

$$r_1 = R + r \tag{12}$$

Alternatively,

$$\frac{r_1}{\sin(180^\circ - \phi)} = \frac{R}{\sin\theta_1} \quad \text{and} \quad \frac{r_2}{\sin\phi} = \frac{r}{\sin\theta_2}$$
$$\frac{\sin\theta_1}{\sin\theta_2} = \frac{R}{r} \times \frac{r_2}{r_1} = \frac{m}{M} \times \frac{r_2}{r_1}$$

Combining with Eq. (5) gives $r_1 = r_2$

Hence, it is an equilateral triangle with

$$\begin{aligned} \psi_1 &= 60^\circ \\ \psi_2 &= 60^\circ \end{aligned} \tag{13}$$

The distance ρ is calculated from the Cosine Rule.

$$\begin{aligned} \rho^2 &= r^2 + (R+r)^2 - 2r(R+r)\cos 60^\circ \\ \rho &= \sqrt{r^2 + rR + R^2} \end{aligned} \tag{14}$$

Alternative Solution to 1.2

Since μ is infinitesimal, it has no gravitational influences on the motion of neither M nor m .For μ to remain stationary relative to both M and m we must have:

$$\frac{GM\mu}{r_1^2}\cos\theta_1 + \frac{Gm\mu}{r_2^2}\cos\theta_2 = \mu\omega^2\rho = \frac{G(M+m)\mu}{(R+r)^3}\rho \tag{4}$$

$$\frac{GM\mu}{r_1^2}\sin\theta_1 = \frac{Gm\mu}{r_2^2}\sin\theta_2 \tag{5}$$

Note that

$$\frac{r_1}{\sin(180^\circ - \phi)} = \frac{R}{\sin\theta_1}$$
$$\frac{r_2}{\sin\phi} = \frac{r}{\sin\theta_2} \qquad \text{(see figure)}$$
$$\frac{\sin\theta_1}{\sin\theta_2} = \frac{R}{r} \times \frac{r_2}{r_1} = \frac{m}{M} \times \frac{r_2}{r_1} \tag{6}$$

Equations (5) and (6):

$$r_1 = r_2 \tag{7}$$
$$\frac{\sin\theta_1}{\sin\theta_2} = \frac{m}{M} \tag{8}$$
$$\psi_1 = \psi_2 \tag{9}$$

The equation (4) then becomes:

$$M\cos\theta_1 + m\cos\theta_2 = \frac{(M+m)}{(R+r)^3} r_1^2 \rho \tag{10}$$

Equations (8) and (10):

$$\sin(\theta_1 + \theta_2) = \frac{M+m}{M}\frac{r_1^2\rho}{(R+r)^3}\sin\theta_2 \tag{11}$$

Note that from figure,

$$\frac{\rho}{\sin\psi_2} = \frac{r}{\sin\theta_2} \tag{12}$$

Equations (11) and (12):

$$\sin(\theta_1+\theta_2) = \frac{M+m}{M}\frac{r_1^2 r}{(R+r)^3}\sin\psi_2 \tag{13}$$

Also from figure,

$$(R+r)^2 = r_2^2 - 2r_1r_2\cos(\theta_1+\theta_2) + r_1^2 = 2r_1^2\left[1-\cos(\theta_1+\theta_2)\right] \tag{14}$$

Equations (13) and (14):

$$\sin(\theta_1+\theta_2) = \frac{\sin\psi_2}{2\left[1-\cos(\theta_1+\theta_2)\right]} \tag{15}$$

$$\theta_1+\theta_2 = 180^\circ - \psi_1 - \psi_2 = 180^\circ - 2\psi_2 \qquad \text{(see figure)}$$

$\therefore \cos\psi_2 = \frac{1}{2},\ \psi_2 = 60^\circ,\ \psi_1 = 60^\circ$

Hence M and m from an equilateral triangle of sides $(R+r)$

Distance μ to M is $R+r$

Distance μ to m is $R+r$

Distance μ to O is

$$\rho = \sqrt{\left(\frac{R+r}{2}-R\right)^2 + \left\{(R+r)\frac{\sqrt{3}}{2}\right\}^2} = \sqrt{R^2+Rr+r^2}$$

1.3 The energy of the mass μ is given by

$$E = -\frac{GM\mu}{r_1} - \frac{Gm\mu}{r_2} + \tfrac{1}{2}\mu\left(\left(\frac{d\rho}{dt}\right)^2 + \rho^2\omega^2\right) \tag{15}$$

Since the perturbation is in the radial direction, angular momentum is conserved ($r_1 = r_2 = \mathfrak{R}$ and $m = M$),

$$E = -\frac{2GM\mu}{\mathfrak{R}} + \tfrac{1}{2}\mu\left(\left(\frac{d\rho}{dt}\right)^2 + \frac{\rho_0^4\omega_0^2}{\rho^2}\right) \tag{16}$$

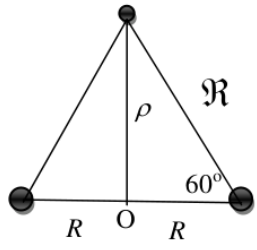
Since the energy is conserved,

$$\frac{dE}{dt} = 0$$

$$\frac{dE}{dt} = \frac{2GM\mu}{\mathfrak{R}^2}\frac{d\mathfrak{R}}{dt} + \mu\frac{d\rho}{dt}\frac{d^2\rho}{dt^2} - \mu\frac{\rho_0^4\omega_0^2}{\rho^3}\frac{d\rho}{dt} = 0 \tag{17}$$

$$\frac{d\mathfrak{R}}{dt} = \frac{d\mathfrak{R}}{d\rho}\frac{d\rho}{dt} = \frac{d\rho}{dt}\frac{\rho}{\mathfrak{R}} \tag{18}$$

$$\frac{dE}{dt} = \frac{2GM\mu}{\mathfrak{R}^3}\rho\frac{d\rho}{dt} + \mu\frac{d\rho}{dt}\frac{d^2\rho}{dt^2} - \mu\frac{\rho_0^4\omega_0^2}{\rho^3}\frac{d\rho}{dt} = 0 \tag{19}$$



Since $\frac{d\rho}{dt} \neq 0$, we have

$$\frac{2GM}{\Re^3}\rho + \frac{d^2\rho}{dt^2} - \frac{\rho_0^4\omega_0^2}{\rho^3} = 0 \quad \text{or}$$

$$\frac{d^2\rho}{dt^2} = -\frac{2GM}{\Re^3}\rho + \frac{\rho_0^4\omega_0^2}{\rho^3}. \tag{20}$$

The perturbation from $\Re_0$ and ρ_0 gives $\Re = \Re_0\left(1 + \frac{\Delta\Re}{\Re_0}\right)$ and $\rho = \rho_0\left(1 + \frac{\Delta\rho}{\rho_0}\right)$.

Then

$$\frac{d^2\rho}{dt^2} = \frac{d^2}{dt^2}(\rho_0 + \Delta\rho) = -\frac{2GM}{\Re_0^3\left(1 + \frac{\Delta\Re}{\Re_0}\right)^3}\rho_0\left(1 + \frac{\Delta\rho}{\rho_0}\right) + \frac{\rho_0^4\omega_0^2}{\rho_0^3\left(1 + \frac{\Delta\rho}{\rho_0}\right)^3} \tag{21}$$

Using binomial expansion $(1+\varepsilon)^n \approx 1 + n\varepsilon$,

$$\frac{d^2\Delta\rho}{dt^2} = -\frac{2GM}{\Re_0^3}\rho_0\left(1 + \frac{\Delta\rho}{\rho_0}\right)\left(1 - \frac{3\Delta\Re}{\Re_0}\right) + \rho_0\omega_0^2\left(1 - \frac{3\Delta\rho}{\rho_0}\right). \tag{22}$$

Using $\Delta\rho = \frac{\Re}{\rho}\Delta\Re$,

$$\frac{d^2\Delta\rho}{dt^2} = -\frac{2GM}{\Re_0^3}\rho_0\left(1 + \frac{\Delta\rho}{\rho_0} - \frac{3\rho_0\Delta\rho}{\Re_0^2}\right) + \rho_0\omega_0^2\left(1 - \frac{3\Delta\rho}{\rho_0}\right). \tag{23}$$

Since $\omega_0^2 = \frac{2GM}{\Re_0^3}$,

$$\frac{d^2\Delta\rho}{dt^2} = -\omega_0^2\rho_0\left(1 + \frac{\Delta\rho}{\rho_0} - \frac{3\rho_0\Delta\rho}{\Re_0^2}\right) + \omega_0^2\rho_0\left(1 - \frac{3\Delta\rho}{\rho_0}\right) \tag{24}$$

$$\frac{d^2\Delta\rho}{dt^2} = -\omega_0^2\rho_0\left(\frac{4\Delta\rho}{\rho_0} - \frac{3\rho_0\Delta\rho}{\Re_0^2}\right) \tag{25}$$

$$\frac{d^2\Delta\rho}{dt^2} = -\omega_0^2\Delta\rho\left(4 - \frac{3\rho_0^2}{\Re_0^2}\right) \tag{26}$$

From the figure, $\rho_0 = \Re_0\cos 30^\circ$ or $\frac{\rho_0^2}{\Re_0^2} = \frac{3}{4}$,

$$\frac{d^2\Delta\rho}{dt^2} = -\omega_0^2\Delta\rho\left(4 - \frac{9}{4}\right) = -\frac{7}{4}\omega_0^2\Delta\rho. \tag{27}$$

Angular frequency of oscillation is $\frac{\sqrt{7}}{2}\,\omega_0$.

Alternative solution:

$M = m$ gives $R = r$ and $\omega_0^2 = \frac{G(M+M)}{(R+R)^3} = \frac{GM}{4R^3}$. The unperturbed radial distance of μ is $\sqrt{3}R$, so the perturbed radial distance can be represented by $\sqrt{3}R + \zeta$ where $\zeta << \sqrt{3}R$ as shown in the following figure.

Using Newton's 2nd law,

$$-\frac{2GM\mu}{\left\{R^2 + (\sqrt{3}R + \zeta)^2\right\}^{3/2}}(\sqrt{3}R + \zeta) = \mu\frac{d^2}{dt^2}(\sqrt{3}R + \zeta) - \mu\omega^2(\sqrt{3}R + \zeta). \tag{1}$$

The conservation of angular momentum gives

$$\mu\omega_0(\sqrt{3}R)^2 = \mu\omega(\sqrt{3}R + \zeta)^2. \tag{2}$$

Manipulate (1) and (2) algebraically, applying $\zeta^2 \approx 0$ and binomial approximation.

$$-\frac{2GM}{\left\{R^2 + (\sqrt{3}R + \zeta)^2\right\}^{3/2}}(\sqrt{3}R + \zeta) = \frac{d^2\zeta}{dt^2} - \frac{\omega_0^2\sqrt{3}R}{(1 + \zeta/\sqrt{3}R)^3}$$

$$-\frac{2GM}{\left\{4R^2+2\sqrt{3}\zeta R\right\}^{3/2}}(\sqrt{3}R+\zeta)\approx\frac{d^2\zeta}{dt^2}-\frac{\omega_0^2\sqrt{3}R}{(1+\zeta/\sqrt{3}R)^3}$$

$$-\frac{GM}{4R^3}\sqrt{3}R\frac{(1+\zeta/\sqrt{3}R)}{(1+\sqrt{3}\zeta/2R)^{3/2}}=\frac{d^2\zeta}{dt^2}-\frac{\omega_0^2\sqrt{3}R}{(1+\zeta/\sqrt{3}R)^3}$$

$$-\omega_0^2\sqrt{3}R\left(1-\frac{3\sqrt{3}\zeta}{4R}\right)\left(1+\frac{\zeta}{\sqrt{3}R}\right)\approx\frac{d^2\zeta}{dt^2}-\omega_0^2\sqrt{3}R\left(1-\frac{3\zeta}{\sqrt{3}R}\right)$$

$$\frac{d^2}{dt^2}\zeta=-\left(\frac{7}{4}\omega_0^2\right)\zeta$$

1.4 Relative velocity

Let v = speed of each spacecraft as it moves in circle around the centre O. The relative velocities are denoted by the subscripts A, B and C. For example, $v_{\rm BA}$ is the velocity of B as observed by A.

The period of circular motion is 1 year

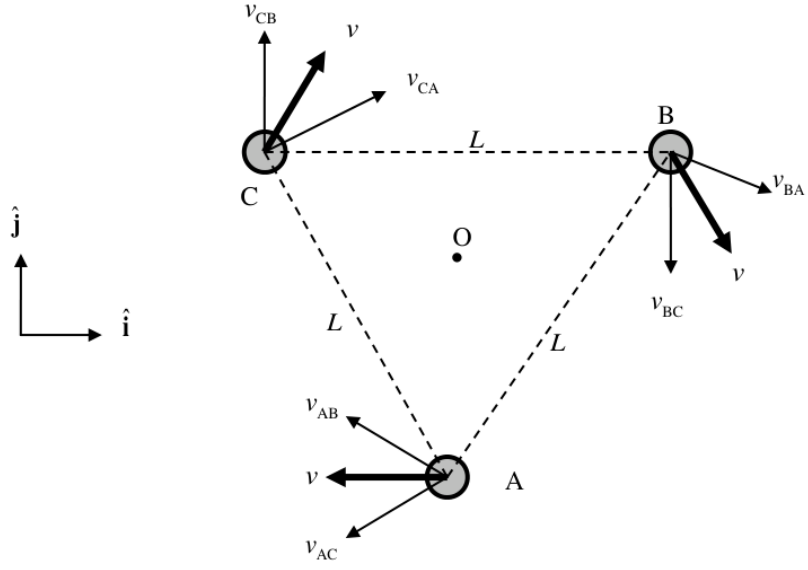
$$T = 365\times 24\times 60\times 60 \text{ s}. \tag{28}$$

The angular frequency $\omega=\frac{2\pi}{T}$

$$\text{The speed } v=\omega\frac{L}{2\cos 30^\circ}=575 \text{ m/s} \tag{29}$$

The speed is much less than the speed light $\rightarrow$ Galilean transformation.

In Cartesian coordinates, the velocities of B and C (as observed by O) are



For B, $\vec{v}_B = v\cos 60^\circ\hat{\mathbf{i}} - v\sin 60^\circ\hat{\mathbf{j}}$

For C, $\vec{v}_C = v\cos 60^\circ\hat{\mathbf{i}} + v\sin 60^\circ\hat{\mathbf{j}}$

Hence $\vec{v}_{\rm BC} = -2v\sin 60^\circ\hat{\mathbf{j}} = -\sqrt{3}v\hat{\mathbf{j}}$

$$\text{The speed of B as observed by C is } \sqrt{3}v\approx 996 \text{ m/s}. \tag{30}$$

Notice that the relative velocities for each pair are anti-parallel.

Alternative solution for 1.4

One can obtain $v_{\rm BC}$ by considering the rotation about the axis at one of the spacecrafts.

$$v_{\rm BC}=\omega L=\frac{2\pi}{365\times 24\times 60\times 60 \text{ s}}\,(5\times 10^6 \text{ km})\approx 996 \text{ m/s}$$