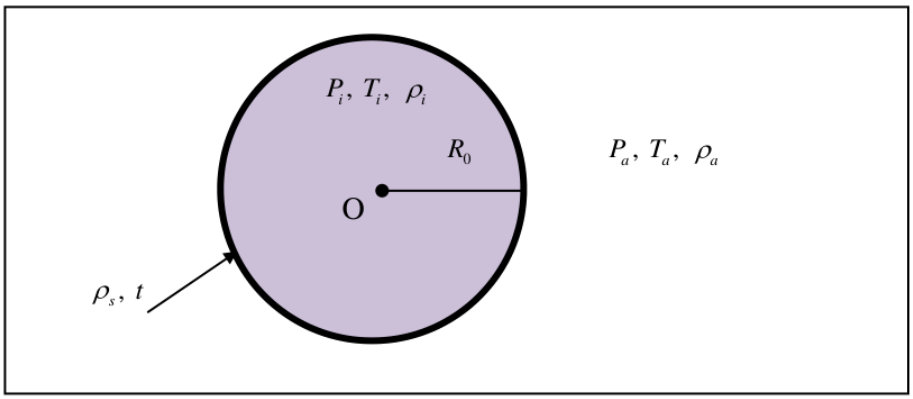
## 2. SOLUTION

2.1. The bubble is surrounded by air.




Cutting the sphere in half and using the projected area to balance the forces give

$$\begin{aligned} P_i \pi R_0^2 &= P_a \pi R_0^2 + 2\,(2\pi R_0 \gamma) \\ P_i &= P_a + \frac{4\gamma}{R_0} \end{aligned} \tag{1}$$

The pressure and density are related by the ideal gas law:

$$PV = nRT \quad \text{or} \quad P = \frac{\rho RT}{M}, \quad \text{where } M = \text{the molar mass of air.} \tag{2}$$

Apply the ideal gas law to the air inside and outside the bubble, we get

$$\begin{aligned} \rho_i T_i &= P_i \frac{M}{R} \\ \rho_a T_a &= P_a \frac{M}{R}, \\ \frac{\rho_i T_i}{\rho_a T_a} &= \frac{P_i}{P_a} = \left[1 + \frac{4\gamma}{R_0 P_a}\right] \end{aligned} \tag{3}$$

2.2. Using $\gamma = 0.025\,\mathrm{Nm}^{-1}$, $R_0 = 1.0\,\mathrm{cm}$ and $P_a = 1.013 \times 10^5\,\mathrm{Nm}^{-2}$, the numerical value of the ratio is

$$\frac{\rho_i T_i}{\rho_a T_a} = 1 + \frac{4\gamma}{R_0 P_a} = 1 + 0.0001 \tag{4}$$

**(The effect of the surface tension is very small.)**

2.3. Let $W$ = total weight of the bubble, $F$ = buoyant force due to air around the bubble

$$\begin{aligned} W &= (\text{mass of film+mass of air})\, g \\ &= \left(4\pi R_0^2 \rho_s t + \frac{4}{3}\pi R_0^3 \rho_i\right) g \\ &= 4\pi R_0^2 \rho_s t g + \frac{4}{3}\pi R_0^3 \frac{\rho_a T_a}{T_i}\left[1 + \frac{4\gamma}{R_0 P_a}\right] g \end{aligned} \tag{5}$$

The buoyant force due to air around the bubble is

$$B = \frac{4}{3}\pi R_0^3 \rho_a g \tag{6}$$

If the bubble floats in still air,

$$B \geq W$$

$$\frac{4}{3}\pi R_0^3 \rho_a g \geq 4\pi R_0^2 \rho_s t g + \frac{4}{3}\pi R_0^3 \frac{\rho_a T_a}{T_i}\left[1 + \frac{4\gamma}{R_0 P_a}\right] g \tag{7}$$

Rearranging to give

$$T_i \geq \frac{R_0\rho_a T_a}{R_0\rho_a - 3\rho_s t}\left[1 + \frac{4\gamma}{R_0 P_a}\right] \geq 307.1\,\mathrm{K} \tag{8}$$

The air inside must be about 7.1 °C warmer.

2.4. Ignore the radius change → Radius remains $R_0 = 1.0\,\mathrm{cm}$.

**(The radius actually decreases by 0.8% when the temperature decreases from 307.1 K to 300 K. The film itself also becomes slightly thicker.)**

$$\text{The drag force from Stokes' Law is } F = 6\pi\eta R_0 u \tag{9}$$

If the bubble floats in the updraught,

$$F \geq W - B$$

$$6\pi\eta R_0 u \geq \left(4\pi R_0^2\rho_s t + \frac{4}{3}\pi R_0^3\rho_i\right)g - \frac{4}{3}\pi R_0^3\rho_a g \tag{10}$$

When the bubble is in thermal equilibrium $T_i = T_a$.

$$6\pi\eta R_0 u \geq \left(4\pi R_0^2\rho_s t + \frac{4}{3}\pi R_0^3\rho_a\left[1 + \frac{4\gamma}{R_0 P_a}\right]\right)g - \frac{4}{3}\pi R_0^3\rho_a g$$

Rearranging to give

$$u \geq \frac{4R_0\rho_s t g}{6\eta} + \frac{\frac{4}{3}R_0^2\rho_a g\left(\frac{4\gamma}{R_0 P_a}\right)}{6\eta} \tag{11}$$

2.5. The numerical value is $u \geq 0.36\,\mathrm{m/s}$.

**The 2nd term is about 3 orders of magnitude lower than the 1st term.**

**From now on, ignore the surface tension terms.**

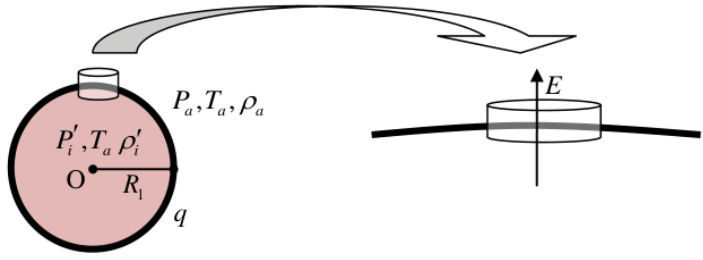
2.6. When the bubble is electrified, the electrical repulsion will cause the bubble to expand in size and thereby raise the buoyant force.

*The force/area is (e-field on the surface × charge/area)*

*There are two alternatives to calculate the electric field ON the surface of the soap film.*

**A. From Gauss's Law**

Consider a very thin pill box on the soap surface.




$E$ = electric field on the film surface that results from all other parts of the soap film, excluding the surface inside the pill box itself.

$$E_q = \text{total field just outside the pill box} = \frac{q}{4\pi\varepsilon_0 R_1^2} = \frac{\sigma}{\varepsilon_0}$$

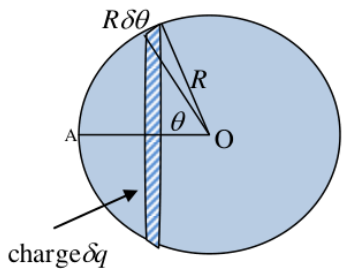
$$= E + \text{electric field from surface charge } \sigma$$

$$= E + E_\sigma$$

Using Gauss's Law on the pill box, we have $E_\sigma = \frac{\sigma}{2\varepsilon_0}$ perpendicular to the film as a result of symmetry.

$$\text{Therefore, } E = E_q - E_\sigma = \frac{\sigma}{\varepsilon_0} - \frac{\sigma}{2\varepsilon_0} = \frac{\sigma}{2\varepsilon_0} = \frac{1}{2\varepsilon_0}\frac{q}{4\pi R_1^2} \tag{12}$$

**B. From direct integration**




$$\delta q = \left(\frac{q}{4\pi R^2}\right) 2\pi R \sin\theta . R\,\delta\theta$$

To find the magnitude of the electrical repulsion we must first find the electric field intensity $E$ at a point on (not outside) the surface itself.

Field at A in the direction $\overrightarrow{OA}$ is

$$\delta E_A = \frac{1}{4\pi\varepsilon_0}\frac{\left(q/4\pi R_1^2\right) 2\pi R_1^2 \sin\theta\,\delta\theta}{\left(2R_1 \sin\frac{\theta}{2}\right)^2} \sin\frac{\theta}{2} = \frac{\left(q/4\pi R_1^2\right)}{2\varepsilon_0} \cos\frac{\theta}{2}\,\delta\left(\frac{\theta}{2}\right)$$

$$E_A = \frac{\left(q/4\pi R_1^2\right)}{2\varepsilon_0}\int_{\theta=0}^{\theta=180^\circ} \cos\frac{\theta}{2}\,d\left(\frac{\theta}{2}\right) = \frac{\left(q/4\pi R_1^2\right)}{2\varepsilon_0} \tag{13}$$

The repulsive force per unit area of the surface of bubble is

$$\left(\frac{q}{4\pi R_1^2}\right) E = \frac{\left(q/4\pi R_1^2\right)^2}{2\varepsilon_0} \tag{14}$$

Let $P_i'$ and $\rho_i'$ be the new pressure and density when the bubble is electrified.

This electric repulsive force will augment the gaseous pressure $P_i'$. $P_i'$ is related to the original $P_i$ through the gas law.

$$P_i' \frac{4}{3}\pi R_1^3 = P_i \frac{4}{3}\pi R_0^3$$

$$P_i' = \left(\frac{R_0}{R_1}\right)^3 P_i = \left(\frac{R_0}{R_1}\right)^3 P_a \tag{15}$$

In the last equation, the surface tension term has been ignored.

From balancing the forces on the half-sphere projected area, we have (again ignoring the surface tension term)

$$\begin{aligned} P_i' + \frac{\left(q/4\pi R_1^2\right)^2}{2\varepsilon_0} &= P_a \\ P_a\left(\frac{R_0}{R_1}\right)^3 + \frac{\left(q/4\pi R_1^2\right)^2}{2\varepsilon_0} &= P_a \end{aligned} \tag{16}$$

Rearranging to get

$$\left(\frac{R_1}{R_0}\right)^4 - \frac{R_1}{R_0} - \frac{q^2}{32\pi^2\varepsilon_0 R_0^4 P_a} = 0 \tag{17}$$

Note that (17) yields $\frac{R_1}{R_0} = 1$ when $q = 0$, as expected.

2.7. Approximate solution for $R_1$ when $\frac{q^2}{32\pi^2\varepsilon_0 R_0^4 P_a} << 1$.

Write $R_1 = R_0 + \Delta R$, $\Delta R << R_0$

$$\text{Therefore, } \frac{R_1}{R_0} = 1 + \frac{\Delta R}{R_0}, \quad \left(\frac{R_1}{R_0}\right)^4 \approx 1 + 4\frac{\Delta R}{R_0} \tag{18}$$

Eq. (17) gives:

$$\Delta R \approx \frac{q^2}{96\pi^2\varepsilon_0 R_0^3 P_a} \tag{19}$$

$$R_1 \approx R_0 + \frac{q^2}{96\pi^2\varepsilon_0 R_0^3 P_a} \approx R_0\left(1 + \frac{q^2}{96\pi^2\varepsilon_0 R_0^4 P_a}\right) \tag{20}$$

2.8. The bubble will float if

$$B \geq W$$

$$\frac{4}{3}\pi R_1^3 \rho_a g \geq 4\pi R_0^2 \rho_s t g + \frac{4}{3}\pi R_0^3 \rho_i g \tag{21}$$

Initially, $T_i = T_a \Rightarrow \rho_i = \rho_a$ for $\gamma \to 0$ and $R_1 = R_0\left(1 + \frac{\Delta R}{R_0}\right)$.

$$\begin{aligned}
\frac{4}{3}\pi R_0^3 \left(1 + \frac{\Delta R}{R_0}\right)^3 \rho_a g &\geq 4\pi R_0^2 \rho_s t g + \frac{4}{3}\pi R_0^3 \rho_a g \\
\frac{4}{3}\pi \left(3\Delta R\right) \rho_a g &\geq 4\pi R_0^2 \rho_s t g \\
\frac{4}{3}\pi \frac{3q^2}{96\pi^2\varepsilon_0 R_0 P_a} \rho_a g &\geq 4\pi R_0^2 \rho_s t g \\
q^2 &\geq \frac{96\pi^2 R_0^3 \rho_s t \varepsilon_0 P_a}{\rho_a}
\end{aligned} \tag{22}$$

$$q \approx 256 \times 10^{-9}\,\text{C} \approx 256\,\text{nC}$$

Note that if the surface tension term is retained, we get

$$R_1 \approx \left(1 + \frac{q^2/96\pi^2\varepsilon_0 R_0^4 P_a}{\left[1 + \frac{2}{3}\left(\frac{4\gamma}{R_0 P_a}\right)\right]}\right) R_0$$