**QUESTION 3: SOLUTION**

1. Using Coulomb's Law, we write the electric field at a distance $r$ is given by

$$E_p = \frac{q}{4\pi\varepsilon_0(r-a)^2} - \frac{q}{4\pi\varepsilon_0(r+a)^2}$$
$$E_p = \frac{q}{4\pi\varepsilon_0 r^2}\left(\frac{1}{\left(1-\frac{a}{r}\right)^2} - \frac{1}{\left(1+\frac{a}{r}\right)^2}\right) \tag{1}$$

Using binomial expansion for small $a$,

$$\begin{aligned} E_p &= \frac{q}{4\pi\varepsilon_0 r^2}\left(1+\frac{2a}{r}-1+\frac{2a}{r}\right) \\ &= +\frac{4qa}{4\pi\varepsilon_0 r^3} = +\frac{qa}{\pi\varepsilon_0 r^3} \\ &= \frac{2p}{4\pi\varepsilon_0 r^3} \end{aligned} \tag{2}$$

2. The electric field seen by the atom from the ion is

$$\vec{E}_{ion} = -\frac{Q}{4\pi\varepsilon_0 r^2}\hat{r} \tag{3}$$

The induced dipole moment is then simply

$$\vec{p} = \alpha\vec{E}_{ion} = -\frac{\alpha Q}{4\pi\varepsilon_0 r^2}\hat{r} \tag{4}$$

From eq. (2)

$$\vec{E}_p = \frac{2p}{4\pi\varepsilon_0 r^3}\hat{r}$$

The electric field intensity $\vec{E}_p$ at the position of an ion at that instant is, using eq. (4),

$$\vec{E}_p = \frac{1}{4\pi\varepsilon_0 r^3}\left[-\frac{2\alpha Q}{4\pi\varepsilon_0 r^2}\hat{r}\right] = -\frac{\alpha Q}{8\pi^2\varepsilon_0^2 r^5}\hat{r}$$

The force acting on the ion is

$$\vec{f} = Q\vec{E}_p = -\frac{\alpha Q^2}{8\pi^2\varepsilon_0^2 r^5}\hat{r} \tag{5}$$

The "-" sign implies that this force is attractive and $Q^2$ implies that the force is attractive regardless of the sign of $Q$.

3. The potential energy of the ion-atom is given by

$$U = \int_r^\infty \vec{f}.d\vec{r} \tag{6}$$

Using this,

$$U = \int_r^\infty \vec{f}.d\vec{r} = -\frac{\alpha Q^2}{32\pi^2\varepsilon_0^2 r^4} \tag{7}$$

[Remark: Students might use the term $-\vec{p}\cdot\vec{E}$ which changes only the factor in front.]

4. At the position $r_{\min}$ we have, according to the Principle of Conservation of Angular Momentum,

$$\begin{aligned} mv_{\max}r_{\min} &= mv_0 b \\ v_{\max} &= v_0\frac{b}{r_{\min}} \end{aligned} \tag{8}$$

And according to the Principle of Conservation of Energy:

$$\frac{1}{2}mv_{\max}^2 + \frac{-\alpha Q^2}{32\pi^2\varepsilon_0^2 r^4} = \frac{1}{2}mv_0^2 \tag{9}$$

Eqs.(12) & (13):

$$\left(\frac{b}{r_{\min}}\right)^2 - \frac{\alpha Q^2/\frac{1}{2}mv_0^2}{32\pi^2\varepsilon_0^2 b^4}\left(\frac{b}{r_{\min}}\right)^4 = 1$$

$$\left(\frac{r_{\min}}{b}\right)^4 - \left(\frac{r_{\min}}{b}\right)^2 + \frac{\alpha Q^2}{16\pi^2\varepsilon_0^2 m v_0^2 b^4} = 0 \tag{10}$$

The roots of eq. (14) are:

$$r_{\min} = \frac{b}{\sqrt{2}}\left[1 \pm \sqrt{1 - \frac{\alpha Q^2}{4\pi^2\varepsilon_0^2 m v_0^2 b^4}}\right]^{\frac{1}{2}} \tag{11}$$

[Note that the equation (14) implies that $r_{\min}$ cannot be zero, unless $b$ is itself zero.] Since the expression has to be valid at $Q = 0$, which gives

$$r_{\min} = \frac{b}{\sqrt{2}}\left[1 \pm 1\right]^{\frac{1}{2}}$$

We have to choose "+" sign to make $r_{\min} = b$

Hence,

$$r_{\min} = \frac{b}{\sqrt{2}}\left[1 + \sqrt{1 - \frac{\alpha Q^2}{4\pi^2\varepsilon_0^2 m v_0^2 b^4}}\right]^{\frac{1}{2}} \tag{12}$$

5. A spiral trajectory occurs when (16) is imaginary (because there is no minimum distance of approach). $r_{\min}$ is real under the condition:

$$1 \geq \frac{\alpha Q^2}{4\pi^2\varepsilon_0^2 m v_0^2 b^4}$$

$$b \geq b_0 = \left(\frac{\alpha Q^2}{4\pi^2\varepsilon_0^2 m v_0^2}\right)^{\frac{1}{4}} \tag{13}$$

For $b < b_0 = \left(\frac{\alpha Q^2}{4\pi^2\varepsilon_0^2 m v_0^2}\right)^{\frac{1}{4}}$ the ion will collide with the atom.

Hence the atom, as seen by the ion, has a cross-sectional area $A$,

$$A = \pi b_0^2 = \pi\left(\frac{\alpha Q^2}{4\pi^2\varepsilon_0^2 m v_0^2}\right)^{\frac{1}{2}} \tag{14}$$