Problem T1. Focus on sketches (13 points)

Part A. Ballistics (4.5 points)

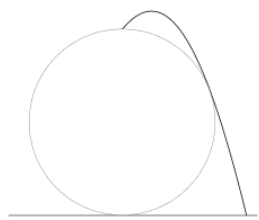
i. (0.8 pts) When the stone is thrown vertically upwards, it can reach the point $x = 0$, $z = v_0^2/2g$ (as it follows from the energy conservation law). Comparing this with the inequality $z \le z_0 - kx^2$ we conclude that

$$z_0 = v_0^2/2g. \qquad [0.3 \text{ pts}]$$

Let us consider the asymptotics $z \to -\infty$; the trajectory of the stone is a parabola, and at this limit, the horizontal displacement (for the given z) is very sensitive with respect to the curvature of the parabola: the flatter the parabola, the larger the displacement. The parabola has the flattest shape when the stone is thrown horizontally, $x = v_0 t$ and $z = -gt^2/2$, i.e. its trajectory is given by $z = -gx^2/2v_0^2$. Now, let us recall that $z \le z_0 - kx^2$, i.e. $-gx^2/2v_0^2 \le z_0 - kx^2 \Rightarrow k \le g/2v_0^2$. Note that $k < g/2v_0^2$ would imply that there is a gap between the parabolic region $z \le z_0 - kx^2$ and the given trajectory $z = -gx^2/2v_0^2$. This trajectory is supposed to be optimal for hitting targets far below ($z \to -\infty$), so there should be no such a gap, and hence, we can exclude the option $k < g/2v_0^2$. This leaves us with

$$k = g/2v_0^2. \qquad [0.5 \text{ pts}]$$

ii. (1.2 pts) Let us note that the stone trajectory is reversible and due to the energy conservation law, one can equivalently ask, what is the minimal initial speed needed for a stone to be thrown from the topmost point of the spherical building down to the ground without hitting the roof, and what is the respective trajectory. It is easy to understand that the trajectory either needs to touch the roof, or start horizontally from the topmost point with the curvature radius equal to R. Indeed, if neither were the case, it would be possible to keep the same throwing angle and just reduce the speed a little bit — the stone would still reach the ground without hitting the roof. Further, if it were tangent at the topmost point, the trajectory wouldn't touch nor intersect the roof anywhere else, because the curvature of the parabola has maximum at its topmost point. Then, it would be possible to keep the initial speed constant, and increase slightly the throwing angle (from horizontal to slightly upwards): the new trajectory wouldn't be neither tangent at the top nor touch the roof at any other point; now we can reduce the initial speed as we argued previously. So we conclude that the optimal trajectory needs to touch the roof somewhere, as shown in Fig.



iii. (2.5 pts) The brute force approach would be writing down the condition that the optimal trajectory intersects with the building at two points and touches at one. This would be described by a fourth order algebraic equation and therefore, it is not realistic to accomplish such a solution within a reasonable time frame.

Note that the interior of the building needs to lie inside the region where the targets can be hit with a stone thrown from the top with initial speed $v_{\min}$. Indeed, if we can throw over the building, we can hit anything inside by lowering the throwing angle. On the other hand, the boundary of the targetable region needs to touch the building. Indeed, if there were a gap, it would be possible to hit a target just above the point where the optimal trajectory touches the building; the trajectory through that target wouldn't touch the building anywhere, hence we arrive at a contradiction.

So, with v_0 corresponding to the optimal trajectory, the targetable region touches the building; due to symmetry, overall there are two touching points (for smaller speeds, there would be four, and for larger speeds, there would be none). With the origin at the top of the building, the intersection points are defined by the following system of equations:

$$x^2 + z^2 + 2zR = 0, \qquad z = \frac{v_0^2}{2g} - \frac{gx^2}{2v_0^2}.$$

Upon eliminating z, this becomes a biquadratic equation for x:

$$x^4 \left(\frac{g}{2v_0^2}\right)^2 + x^2 \left(\frac{1}{2} - \frac{gR}{v_0^2}\right) + \left(\frac{v_0^2}{4g} + R\right) \frac{v_0^2}{g} = 0.$$

Hence the speed by which the real-valued solutions disappear can be found from the condition that the discriminant vanishes:

$$\left(\frac{1}{2} - \frac{gR}{v_0^2}\right)^2 = \frac{1}{4} + \frac{gR}{v_0^2} \implies \frac{gR}{v_0^2} = 2.$$

Bearing in mind that due to the energy conservation law, at the ground level the squared speed is increased by $4gR$. Thus we finally obtain

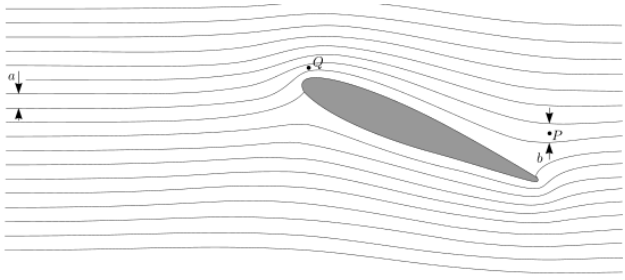
$$v_{\min} = \sqrt{v_0^2 + 4gR} = 3\sqrt{\frac{gR}{2}}.$$

Part B. Mist (4 points)

i. (0.8 pts) In the plane's reference frame, along the channel between two streamlines the volume flux of air (volume flow rate) is constant due to continuity. The volume flux is the product of speed and channel's cross-section area, which, due to the two-dimensional geometry, is proportional to the channel width and can be measured from the Fig. Due to the absence of wind, the unperturbed air's speed in the plane's frame is just v_0. So, upon measuring the dimensions $a = 10\,\text{mm}$ and $b = 13\,\text{mm}$ (see Fig), we can write $v_0 a = ub$ and hence $u = v_0 \frac{a}{b}$. Since at point P, the streamlines are horizontal where all the velocities are parallel, the vector addition is reduced to the scalar addition: the air's ground speed

$$v_P = v_0 - u = v_0 \left(1 - \frac{a}{b}\right) = 23\,\text{m/s}.$$

ii. (1.2 pts) Although the dynamic pressure $\frac{1}{2}\rho v^2$ is relatively small, it gives rise to some adiabatic expansion and compression. In expanding regions the temperature will drop and hence, the pressure of saturated vapours will also drop. If the dew point is reached, a stream of droplets will appear. This process will start in a point where the adiabatic expansion is maximal, i.e. where the hydrostatic pressure is minimal and consequently, as it follows from the Bernoulli's law $p + \frac{1}{2}\rho v^2 = \text{const}$, the dynamic pressure is maximal: in the place where the air speed in wing's frame is maximal and the streamline distance minimal. Such a point Q is marked in Fig.



iii. (2 pts) First we need to calculate the dew point for the air of given water content (since the relative pressure change will be small, we can ignore the dependence of the

dew point on pressure). The water vapour pressure is $p_w = p_{sa}\, r = 2.08\,\mathrm{kPa}$. The relative change of the pressure of the saturated vapour is small, so we can linearize its temperature dependence:

$$\frac{p_{sa} - p_w}{T_a - T} = \frac{p_{sb} - p_{sa}}{T_b - T_a} \implies T_a - T = (T_b - T_a)\frac{(1-r)p_{sa}}{p_{sb} - p_{sa}};$$

numerically $T \approx 291.5\,\mathrm{K}$. Further we need to relate the air speed to the temperature. To this end we need to use the energy conservation law. A convenient ready-to-use form of it is provided by the Bernoulli's law. Applying this law will give a good approximation of the reality, but strictly speaking, it needs to be modified to take into account the compressibility of air and the associated expansion/contraction work. Consider one mole of air, which has the mass μ and the volume $V = RT/p$. Apparently the process is fast and the air parcels are large, so that heat transfer across the air parcels is negligible. Additionally, the process is subsonic; all together we can conclude that the process is adiabatic. Consider a segment of a tube formed by the streamlines. Let us denote the physical quantities at its one end by index 1, and at the other end — by index 2. Then, while one mole of gas flows into the tube at one end, as much flows out at the other end. The inflow carries in kinetic energy $\frac{1}{2}\mu v_1^2$, and the outflow carries out $\frac{1}{2}\mu v_2^2$. The inflowing gas receives work due to the pushing gas equal to $p_1 V_1 = RT_1$, the outflowing gas performs work $p_2 V_2 = RT_2$. Let's define molar heat capacities $C_V = \mu c_V$ and $C_p = \mu c_p$. The inflow carries in heat energy $C_V RT_1$, and the outflow carries out $C_V RT_2$. All together, the energy balance can be written as $\frac{1}{2}\mu v^2 + C_p T = \mathrm{const}$. From this we can easily express

$$\Delta\frac{v^2}{2} = \frac{1}{C}v_{\mathrm{crit}}^2\left(\frac{a^2}{c^2} - 1\right) = c_p\,\Delta T,$$

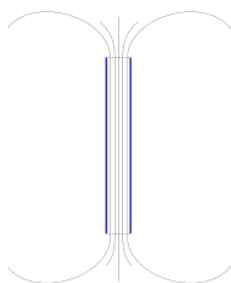
where c is the streamline distance at the point Q, and further

$$v_{\mathrm{crit}} = c\sqrt{\frac{2c_p\Delta T}{a^2 - c^2}} \approx 23\,\mathrm{m/s},$$

where we have used $c \approx 4.5\,\mathrm{mm}$ and $\Delta T = 1.5\,\mathrm{K}$. Note that in reality, the required speed is probably somewhat higher, because for a fast condensation, a considerable over-saturation is needed. However, within an order of magnitude, this estimate remains valid.

Part C. Magnetic straws (4.5 points)

i. (0.8 pts) Due to the superconducting walls, the magnetic field lines cannot cross the walls, so the flux is constant along the tube. For a closed contour inside the tube, there should be no circulation of the magnetic field, hence the field lines cannot be curved, and the field needs to be homogeneous. The field lines close from outside the tube, similarly to a solenoid.



ii. (1.2 pts) Let us consider the change of the magnetic energy when the tube is stretched (virtually) by a small amount Δl. Note that the magnetic flux trough the tube is conserved: any change of flux would imply a non-zero electromotive force $\frac{d\Phi}{dt}$, and for a zero resistivity, an infinite current. So, the induction $B = \frac{\Phi}{\pi r^2}$. The energy density of the magnetic field is $\frac{B^2}{2\mu_0}$. Thus, the change of the magnetic energy is calculated as

$$\Delta W = \frac{B^2}{2\mu_0}\pi r^2\,\Delta l = \frac{\Phi^2}{2\mu_0\pi r^2}\,\Delta l.$$

This energy increase is achieved owing to the work done by the stretching force, $\Delta W = T\,\Delta l$. Hence, the force

$$T = \frac{\Phi^2}{2\mu_0 \pi r^2}.$$

iii. (2.5 pts) Let us analyse, what would be the change of the magnetic energy when one of the straws is displaced to a small distance. The magnetic field inside the tubes will remain constant due to the conservation of magnetic flux, but outside, the magnetic field will be changed. The magnetic field outside the straws is defined by the following condition: there is no circulation of $\vec{B}$ (because there are no currents outside the straws); there are no sources of the field lines, other than the endpoints of the straws; each of the endpoints of the straws is a source of streamlines with a fixed magnetic flux $\pm\Phi$. These are exactly the same condition as those which define the electric field of four charges $\pm Q$. We know that if the distance between charges is much larger than the geometrical size of a charge, the charges can be considered as point charges (the electric field near the charges remains almost constant, so that the respective contribution to the change of the overall electric field energy is negligible). Therefore we can conclude that the endpoints of the straws can be considered as magnetic point charges. In order to calculate the force between two magnetic charges (magnetic monopoles), we need to establish the correspondence between magnetic and electric quantities.

For two electric charges Q separated by a distance a, the force is $F = \frac{1}{4\pi\varepsilon_0}\frac{Q^2}{a^2}$, and at the position of one charge, the electric field of the other charge has energy density $w = \frac{1}{32\pi^2\varepsilon_0}\frac{Q^2}{a^4}$; hence we can write $F = 8\pi w a^2$. This is a universal expression for the force (for the case when the field lines have the same shape as in the case of two opposite and equal by modulus electric charges) relying only on the energy density, and not related to the nature of the field; so we can apply it to the magnetic field. Indeed, the force can be calculated as a derivative of the full field energy with respect to a virtual displacement of a field line source (electric or magnetic charge); if the energy densities of two fields are respectively equal at one point, they are equal everywhere, and so are equal the full field energies. As it follows from the Gauss law, for a point source of a fixed magnetic flux Φ at a distance a, the induction $B = \frac{1}{4\pi}\frac{\Phi}{a^2}$. So, the energy density $w = \frac{B^2}{2\mu_0} = \frac{1}{32\pi^2\mu_0}\frac{\Phi^2}{a^4}$, hence

$$F = \frac{1}{4\pi\mu_0}\frac{\Phi^2}{a^2}.$$

For the two straws, we have four magnetic charges. The longitudinal (along a straw axis) forces cancel out (the diagonally positioned pairs of same-sign-charges push in opposite directions). The normal force is a superposition of the attraction due to the two pairs of opposite charges, $F_1 = \frac{1}{4\pi\mu_0}\frac{\Phi^2}{l^2}$, and the repulsive forces of diagonal pairs, $F_2 = \frac{\sqrt{2}}{8\pi\mu_0}\frac{\Phi^2}{2l^2}$. The net attractive force will be

$$F = 2(F_1 - F_2) = \frac{4-\sqrt{2}}{8\pi\mu_0}\frac{\Phi^2}{l^2}.$$

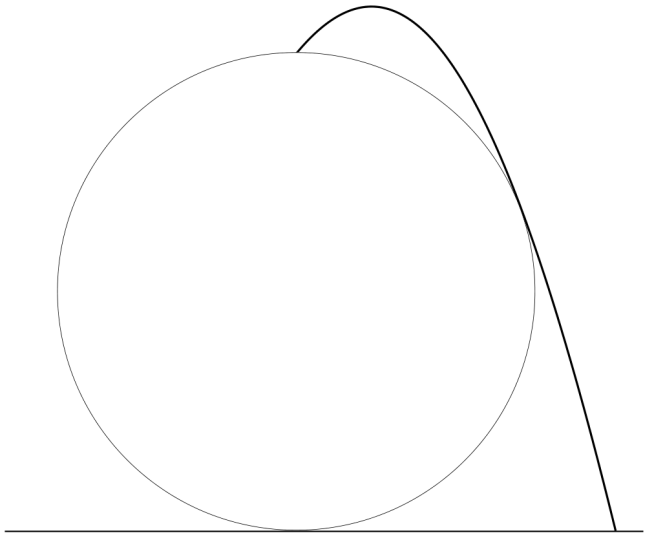
Problem T1. Focus on sketches (13 points)

Part A. Ballistics (4.5 points)

i. (0.8 pts)

$$z_0 = v_0^2/2g$$
$$k = g/2v_0^2$$

ii. (1.2 pts) The sketch of the trajectory:



iii. (2.5 pts)

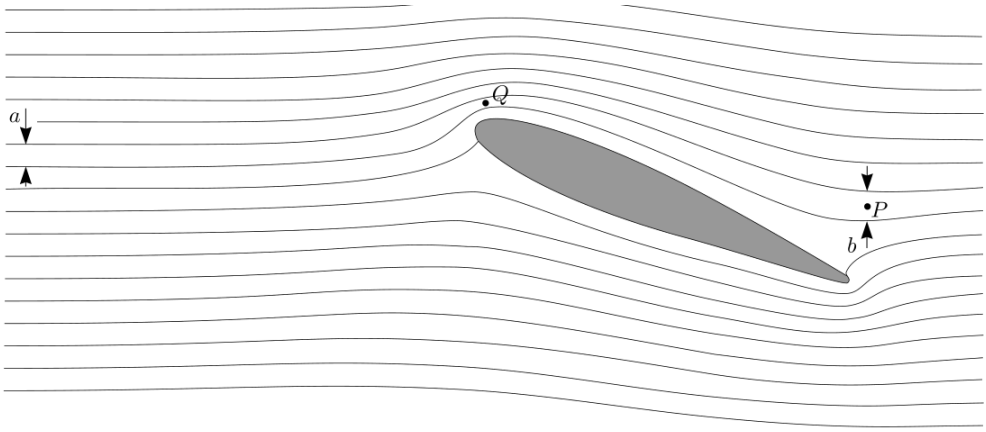
$$v_{\min} = 3\sqrt{\frac{gR}{2}}$$

Part B. Air flow around a wing (4 points)

i. (0.8 pts)

$$v_P = 23\,\mathrm{m/s}$$

ii. (1.2 pts) Mark on this fig. the point Q. Use it also for taking measurements (questions i and iii).



Formulae motivating the choice of point Q:

$$av = \text{const}$$

$$p + \frac{1}{2}\rho v^2 = \text{const}$$

$$p^{1-\gamma}T^{\gamma} = \text{const}$$

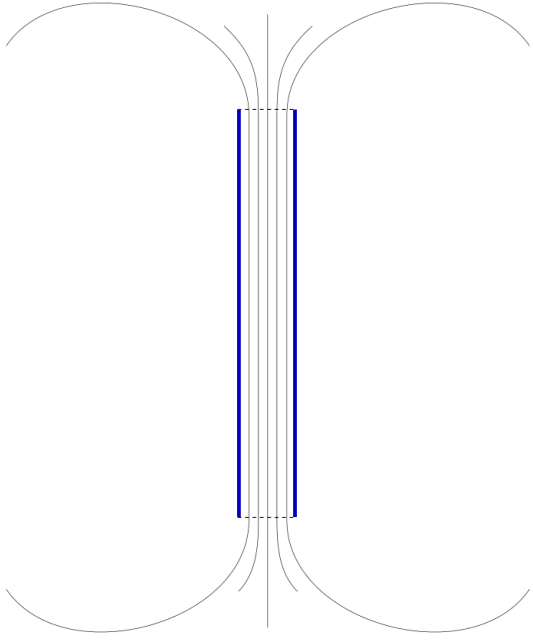
iii. (2.0 pts)

Formula: $v_{\text{crit}} = c\sqrt{\dfrac{2c_p\Delta T}{a^2 - c^2}}$

Numerical: $v_{\text{crit}} \approx 23\,\mathrm{m/s}$

Part C. Magnetic straws (4.5 points)

i. (0.8 pts) Sketch here five magnetic field lines.



ii. (1.2 pts)

$$T = \frac{\Phi^2}{2\mu_0 \pi r^2}$$

iii. (2.5 pts)

$$F = \frac{4 - \sqrt{2}}{8\pi\mu_0} \frac{\Phi^2}{l^2}$$