Cosmic Inflation

A. Expansion of Universe

Question A.1

For any test mass m on the boundary of the sphere, 0.2

$$m\ddot{R}(t) = -GmM_s/R^2(t) \tag{A.1.1}$$

where M_s is mass portion inside the sphere

Multiplying equation (A.1.1) with $\dot{R}$ and integrating it gives 0.6

$$\int \dot{R}\,\frac{d\dot{R}}{dt}\,dt = \frac{1}{2}\dot{R}^2 = \frac{GM_s}{R} + A$$

where A is a integration constant

Taking $M_s = \frac{4}{3}\pi R^3(t)\rho(t)$, and $\dot{R} = \dot{a}\,R_s$ 0.2

$$\left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G}{3}\,\rho(t) + \frac{2A}{R_s^2\,a^2(t)}$$

0.2

Therefore, we have $A_1 = \frac{8\pi G}{3}$ 0.1

Total 1.3

Question A.2

The 2nd Friedmann equation can be obtained from the 1st law of thermodynamics : 0.1

$$dE = -pdV + dQ.$$

For adiabatic processes $dE + pdV = 0$ and its time derivative is $\dot{E} + p\dot{V} = 0$. 0.1

For the sphere $\dot{V} = V\,(3\dot{a}/a)$ 0.1

Its total energy is $E = \rho(t)V(t)c^2$ 0.2

Therefore $\dot{E} = \left(\dot{\rho} + 3\frac{\dot{a}}{a}\right)Vc^2$ 0.1

It yields 0.2

$$\dot{\rho} + 3\left(\rho + \frac{p}{c^2}\right)\frac{\dot{a}}{a} = 0$$

Therefore, we have $A_2 = 3$. 0.1

Total 0.9

Question A.3

Interpreting $\rho(t)c^2$ as total energy density, and substituting $\frac{p(t)}{c^2} = w\,\rho(t)$ in to the 2nd Friedmann equation yields: 0.1

$$\dot{\rho} + 3\rho(1+w)\frac{\dot{a}}{a} = 0$$

$$\rho \propto a^{-3(w+1)}$$

0.2

(i) In case of radiation, photon as example, the energy is given by $E_r = h\nu = hc/\lambda$ then its energy density $\rho_r = \frac{E_r}{V} \propto a^{-4}$ so that $w_r = \frac{1}{3}$ 0.3

(ii) In case of nonrelativistic matter, its energy density nearly $\rho_m \simeq \frac{m_0c^2}{V} \propto a^{-3}$ since dominant energy comes from its rest energy m_0c^2, so that $w_m = 0$ 0.3

(iii) For a constant energy density, let say ϵ_Λ = constant, $\epsilon_\Lambda \propto a^0$ so that $w_\Lambda = -1$. 0.3

Total 1.2

Question A.4

(i) In case of $k = 0$, for radiation we have $\rho_r a^4$ = constant. So by comparing the parameters values with their present value, $\rho_r(t)a^4(t) = \rho_{r0}a_0^4$, 0.2

$$\left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G}{3}\,\rho_{r0}\left(\frac{a_0}{a}\right)^4.$$

$$\int a\,da = \frac{1}{2}a^2 + K = \left(\frac{8\pi G}{3}\,\rho_{r0}a_0^4\right)^{\frac{1}{2}} t.$$

Because $a(t=0) = 0, K = 0$, then 0.2

$$a(t) = (2)^{\frac{1}{2}}\left(\frac{8\pi G}{3}\,\rho_{r0}a_0^4\right)^{\frac{1}{4}} t^{\frac{1}{2}} = (2H_0)^{\frac{1}{2}}\,t^{\frac{1}{2}}.$$

where $H_0 = \left(\frac{8\pi G}{3}\,\rho_{r0}\right)^{\frac{1}{2}}$ after taking $a_0 = 1$.

(ii) for non-relativistic matter domination, using $\rho_m(t)a^3(t) = \rho_{m0}a_0^3$, and similar way we will get 0.4

$$a(t) = \left(\frac{3}{2}\right)^{\frac{2}{3}}\left(\frac{8\pi G}{3}\,\rho_{m0}a_0^4\right)^{\frac{1}{3}} t^{\frac{2}{3}} = \left(\frac{3H_0}{2}\right)^{\frac{2}{3}} t^{\frac{2}{3}}.$$

where $H_0 = \left(\frac{8\pi G}{3}\,\rho_{m0}\right)^{\frac{1}{2}}$.

(iii) for constant energy density, 0.4

$$\ln a = H_0 t + K'$$

Where K' is integration constant and $H_0 = \left(\frac{8\pi G}{3}\,\rho_\Lambda\right)^{\frac{1}{2}}$. Taking condition $a_0 = 1$,

$$\ln\left(\frac{a}{a_0}\right) = H_0(t - t_0)$$

$$a(t) = e^{H_0(t-t_0)}$$

Total 1.2

Question A.5

Condition for critical energy condition: 0.1

$$\rho_c(t) = \frac{3H^2}{8\pi G}$$

Friedmann equation can be written as

$$H^2(t) = H^2(\mathrm{t})\,\Omega(\mathrm{t}) - \frac{kc^2}{R_0^2\,a^2(t)}$$

$$\left(\frac{R_0^2}{\mathrm{c}^2}\right) a^2H^2(\Omega - 1) = k \qquad \text{(A.5.1)}$$

Total 0.1

Question A.6

Because $\left(\frac{R_0^2}{\mathrm{c}^2}\right) a^2H^2 > 0$, then $k = +1$ corresponds to $\Omega > 1$, $k = -1$ corresponds to $\Omega < 1$ and $k = 0$ corresponds to $\Omega = 1$ 0.3

Total 0.3

B. Motivation To Introduce Inflation Phase and Its General Conditions

Question B.1

Equation (A.5.1) shows that 0.1

$$(\Omega - 1) = \frac{kc^2}{R_0^2}\,\frac{1}{\dot{a}^2}.$$

In a universe dominated by non-relativistic matter or radiation, scale factor can be written as a function of time as $a = a_0\left(\frac{t}{t_0}\right)^p$ where $p < 1$ ($p = \frac{1}{2}$ for radiation and $p = \frac{2}{3}$ for non-relativistic matter) 0.2

$$(\Omega - 1) = \tilde{k}\, t^{2(1-p)}$$

0.2

Total 0.5

Question B.2

For a period dominated by constant energy provides the solution $a(t) = e^{Ht}$ so that $\dot{a} = He^{Ht}$ 0.1

$$(\Omega - 1) = \frac{k}{H^2}\, t^{-2Ht}$$

0.2

Total 0.3

Question B.3

Inflation period can be generated by constant energy period, therefore it is a phase where $w = -1$ so that $p = w\rho c^2 = -\rho c^2$ (negative pressure). 0.2

Differentiating Friedmann equation leads to 0.4

$$\dot{a}^2 = \frac{8\pi G}{3}\,\rho a^2 - \frac{kc^2}{R_0^2}$$

$$2\dot{a}\ddot{a} = \frac{8\pi G}{3}\left(\dot{\rho}a^2 + 2\rho a\dot{a}\right) = \frac{8\pi G}{3}\left(-3\left(\rho + \frac{p}{c^2}\right)a\dot{a} + 2\rho a\dot{a}\right).$$

$$\frac{\ddot{a}}{a} = -\frac{4\pi G}{3}\left(\rho + \frac{3p}{c^2}\right)$$

So that because during inflation $p = -\rho c^2$, it is equivalent with condition $\ddot{a} > 0$ (accelerated expansion) 0.1

As a result, $\ddot{a} = d(\dot{a})/dt = d(Ha)/dt > 0$ or $d(Ha)^{-1}/dt < 0$ (shrinking Hubble radius). 0.2

Total 0.9

Question B.4

Inflation condition can be written as $\frac{d(aH)^{-1}}{dt} < 0$, with $H = \dot{a}/a$ as such 0.2

$$\frac{d(aH)^{-1}}{dt} = -\frac{\dot{a}H + a\dot{H}}{(aH)^2} = -\frac{1}{a}(1 - \epsilon) < 0 \Rightarrow \epsilon < 1$$

Total 0.2

C. Inflation Generated by Homogenously Distributed Matter

Question C.1

Differentiating equations (4) and employing equation 4 we can get 0.3

$$2H\dot{H} = \frac{1}{3M_{pl}^2}\left[\dot{\phi}\ddot{\phi} + \left(\frac{\partial V}{\partial \phi}\right)\dot{\phi}\right] = \frac{1}{3M_{pl}^2}\left[-3H\dot{\phi}^2\right]$$

$$\dot{H} = -\frac{1}{2}\frac{\dot{\phi}^2}{M_{pl}^2}$$

Therefore $\epsilon = \frac{1}{2}\frac{\dot{\phi}^2}{M_{pl}^2 H^2}$ 0.1

The inflation can occur when the potential energy dominates the particle's energy ($\dot{\phi}^2 \ll V$) such that $H^2 \approx V/(3M_{pl}^2)$. 0.2

Slow-roll approximation: $3H\dot{\phi} \approx -V'$ 0.1

Implies 0.3

$$\epsilon \approx \frac{M_{pl}^2}{2}\left(\frac{V'}{V}\right)^2 \tag{C.1.1}$$

we also have 0.4

$$3\dot{H}\dot{\phi} + 3H\ddot{\phi} = -V''\dot{\phi}$$

$$\delta = -\frac{\ddot{\phi}}{H\dot{\phi}} = \frac{V''}{3H^2} - \epsilon$$

Therefore

$$\eta_V \approx M_{Pl}^2\frac{V''}{V} \tag{C.1.2}$$

$$dN = H\,dt = \left(\frac{H}{\dot{\phi}}\right)d\phi \approx -\frac{1}{M_{pl}^2}\left(V/V'\right)d\phi \tag{C.1.3}$$

0.3

$$\frac{dN}{d\phi} \approx -\frac{1}{M_{pl}^2}\left(V/V'\right)$$

Total 1.7

D. Inflation with A Simple Potential

Question D.1

Inflation ends at $\epsilon = 1$. Using $V(\phi) = \Lambda^4\left(\phi/M_{pl}\right)^n$ yields 0.5

$$\epsilon = \frac{M_{pl}^2}{2}\left[\frac{n}{\phi_{end}}\right]^2 = 1 \Rightarrow \phi_{end} = \frac{n}{\sqrt{2}}M_{pl}$$

Total 0.5

Question D.2

From equations (C.1.1), (C.1.2) and (C.1.3) we can obtain 0.2

$$N = -\left[\frac{\phi}{M_{pl}}\right]^2\frac{1}{2n} + \beta$$

where β is a integration constant. As $N = 0$ at ϕ_{end} then $\beta = \frac{n}{4}$.

$$N = -\left[\frac{\phi}{M_{pl}}\right]^2\frac{1}{2n} + \frac{n}{4}$$

$$\eta_V = n(n-1)\left[\frac{M_{pl}}{\phi}\right]^2 = \frac{2(n-1)}{n-4N}$$

0.2

$$\varepsilon = \frac{n^2}{2}\left[\frac{M_{pl}}{\phi}\right]^2 = \frac{n}{n-4N}$$

0.2

so that

$$r = 16\varepsilon = \frac{16n}{n-4N}$$

0.1

$$n_s = 1 + 2\eta_V - 6\epsilon = 1 - \frac{2(n+2)}{(n-4N)}$$

0.1

To obtain the observational constraint $n_s = 0.968$ we need $n = -5.93$ which is inconsistent with the condition $r < 0.12$. There is no a closest integer n that can obtains $r < 0.12$. As example, for $n = -6$ leads a contradiction $0 < (-0.27)$ and for $n = -5$ leads a contradiction $0 < (-0.2)$. 0.1

Total 0.9