Thermoacoustic engine – Solution

Part A: Sound wave in a closed tube

A.1. The boundary conditions are: $u(0,t) = u(L,t) = 0$. As a result, $\sin\left(\frac{2\pi}{\lambda}L\right) = 0$, so we get $\lambda_{\max} = 2L$.

A.2. We get

$$V(x,t) = S \cdot (\Delta x + u(x+\Delta x,t) - u(x,t)) = S\Delta x \cdot (1+u') = V_0 + V_0 u'.$$

Thus,

$$V(x,t) = V_0 + akV_0\cos(kx)\cos(\omega t) \quad \Rightarrow \quad V_1(x) = akV_0\cos(kx).$$

A.3. We use Newton's Second Law $\rho_0\ddot{u} = -p'$ to deduce $p' = -\rho_0\ddot{u} = \rho_0 a\omega^2\sin(kx)\cos(\omega t)$, so that

$$p(x,t) = p_0 - a\frac{\omega^2}{k}\rho_0\cos(kx)\cos(\omega t) \quad \Rightarrow \quad p_1(x) = a\frac{\omega^2}{k}\rho_0\cos(kx).$$

A.4. Using $a \ll L$, we obtain $\frac{p_1(x)}{p_0} = \gamma\frac{V_1(x)}{V_0}$. As a result, $\frac{\rho_0}{p_0}\frac{\omega^2}{k} = \gamma\cdot k$, and $c = \sqrt{\frac{\gamma p_0}{\rho_0}}$.

A.5. The relative change in $T(x,t)$ is the sum of the relative changes in $V(x,t)$ and $p(x,t)$. As a result,

$$T_1(x) = \frac{T_0}{p_0}p_1(x) - \frac{T_0}{V_0}V_1(x) = (\gamma-1)\frac{T_0}{V_0}V_1(x) = ak(\gamma-1)T_0\cos(kx).$$

A.6. The movement of the gas parcels inside the tube conveys heat along its boundary. To determine the direction of the convection, we combining the result of Task A.5 and the expression (1) for $u(x,t)$. We see that when $0 < x < \frac{L}{2}$, the gas is colder when the displacement $u(x,t)$ is positive. Likewise, when $\frac{L}{2} < x < L$, the gas is colder when the displacement $u(x,t)$ is negative. Hence, heat flows into the gas near the point B, cooling it down, and out of the gas near the points A and C, heating them up.

Part B: Sound wave amplification induced by external thermal contact

B.1. We get

$$T_{\text{env}}(t) = T_{\text{plate}}\left(x_0 + u(x_0,t)\right) = T_0 - \frac{\tau}{\ell}\cdot u(x_0,t),$$

so that:

$$T_{\text{st}} = \frac{a\tau}{\ell}\sin(kx_0) = \frac{a\tau}{\ell\sqrt{2}}.$$

B.2. The gas will convey heat from the hot reservoir to the cold one if the parcels are colder than the environment when $u(x_0,t) < 0$, and hotter when $u(x_0,t) > 0$. This occurs precisely if

$$T_{\text{st}} > T_1.$$

Plugging in the results of Tasks A.5 and B.1, we get

$$\frac{a\tau_{\text{cr}}}{\ell}\sin(kx_0) = ak(\gamma-1)T_0\cos(kx_0) \quad \Rightarrow \quad \tau_{\text{cr}} = k\ell(\gamma-1)T_0.$$

B.3. Using the first law of thermodynamics, we get

$$\frac{dQ}{dt} = \frac{dE}{dt} + p\frac{dV}{dt}.$$

Plugging in the relation $E = \frac{1}{\gamma-1}pV$, we see that:

$$\frac{dQ}{dt} = \frac{1}{\gamma-1}\frac{d}{dt}(pV) + p\frac{dV}{dt} = \frac{1}{\gamma-1}V\frac{dp}{dt} + \frac{\gamma}{\gamma-1}p\frac{dV}{dt} \approx \frac{1}{\gamma-1}V_0\frac{dp}{dt} + \frac{\gamma}{\gamma-1}p_0\frac{dV}{dt}.$$

B.4. We plug the expression for $\frac{dQ}{dt}$ into the result of Task B.3. This gives:

$$\frac{1}{\gamma-1}V_0\frac{dp}{dt} + \frac{\gamma}{\gamma-1}p_0\frac{dV}{dt} = \beta V_0(T_{\text{st}} - T_1)\cdot\cos(\omega t).$$

We now plug in the data given in equation (6), and get (by considering terms with $\cos(\omega t)$ and $\sin(\omega t)$ separately):

$$\frac{1}{\gamma-1}V_0 p_a \omega + \frac{\gamma}{\gamma-1}p_0 V_a \omega = \beta V_0 (T_{\text{st}} - T_1)$$
$$\frac{1}{\gamma-1}V_0 p_b \omega - \frac{\gamma}{\gamma-1}p_0 V_b \omega = 0$$

and thus, we can already express V_b as

$$V_b = \frac{1}{\gamma} p_b \cdot \frac{V_0}{p_0}.$$

For V_a, we plug in the results of Tasks B.1 and B.2,

$$T_{\text{st}} - T_1 = \frac{a}{\ell\sqrt{2}}(\tau - \tau_{\text{cr}}),$$

giving:

$$V_a = \left(-\frac{1}{\gamma}p_a - \frac{\gamma-1}{\gamma}\frac{\beta}{\omega}\frac{a}{\ell\sqrt{2}}(\tau-\tau_{\text{cr}})\right)\cdot\frac{V_0}{p_0}.$$

B.5. We want to integrate the mechanical work generated, $\int pdV$, and averaging the result over a long time. To do this, we substitute our expressions (6) for the perturbed p and V. Since the average of $\cos(\omega t)\sin(\omega t)$ is 0, and that of $\sin^2(\omega t)$ and $\cos^2(\omega t)$ is $\frac{1}{2}$, we get:

$$\frac{V_0}{S\ell}W_{\text{tot}} = -\pi\cdot(p_a V_b + p_b V_a).$$

Using the result of B.4, we get

$$\frac{V_0}{S\ell}W_{\text{tot}} = \frac{\pi}{\omega}\cdot\frac{\gamma-1}{\gamma}\beta\frac{a}{\ell\sqrt{2}}(\tau-\tau_{\text{cr}})\cdot V_0\frac{p_b}{p_0}.$$

To leading order, p_b is the unperturbed wave $p_b \approx p_1(x_0) = a\frac{\omega^2}{k}\rho_0\cos(kx_0) = ak\gamma p_0\frac{1}{\sqrt{2}}$. Simplifying, we get

$$W_{\text{tot}} = \frac{\pi}{\omega}S\cdot\frac{\gamma-1}{\gamma}\beta\frac{a}{\sqrt{2}}(\tau-\tau_{\text{cr}})\cdot\frac{p_b}{p_0} = \frac{\pi}{2\omega}(\gamma-1)\beta(\tau-\tau_{\text{cr}})ka^2S.$$

B.6. We want to compute the amount of heat convection over one cycle. This means that we need to take the amount of heat moving in or out of the parcel, and weigh it by the position of the parcel at that time. Thus, the total heat conveyed by the parcel, integrated along a cycle, is:

$$Q_{\text{tot}} = \frac{1}{\Delta x}\int\frac{dQ}{dt}u\cdot dt.$$

This expression can be computed to leading order using $\frac{dQ}{dt} = \beta V_0(T_{\text{st}} - T_1)\cdot\cos(\omega t)$ and the unperturbed displacement $u(x_0,t) = \frac{a}{\sqrt{2}}\cos(\omega t)$. This gives

$$Q_{\text{tot}} = \frac{\pi}{\omega}\beta V_0(T_{\text{st}}-T_1)\frac{a}{\sqrt{2}} = \frac{\pi}{\omega}\beta V_0\cdot\frac{a}{\ell\sqrt{2}}(\tau-\tau_{\text{cr}})\cdot\frac{a}{\sqrt{2}} = \frac{\pi}{2\omega}\beta(\tau-\tau_{\text{cr}})\frac{a^2S}{\ell}.$$

B.7. Dividing the results of Tasks B.5 and B.6, we obtain the expression:

$$\eta = \frac{W_{\text{tot}}}{Q_{\text{tot}}} = (\gamma-1)k\ell = \frac{\tau_{\text{cr}}}{T_0} = \frac{\tau_{\text{cr}}}{\tau}\cdot\frac{\tau}{T_0} = \frac{\tau_{\text{cr}}}{\tau}\cdot\eta_c.$$