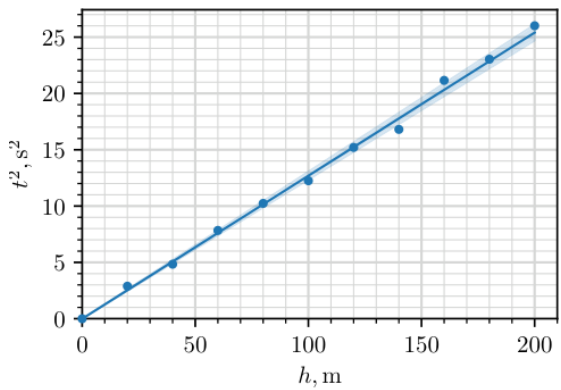
E1: Planet - SOLUTION

A.1: The free-fall acceleration g can be found by dropping the ball from low heights such that the air friction and effects from the curvature of the planet are minimized. We also choose the radius and density of the ball to be as big as possible to minimize the effect of air friction, i.e. setting $r = 50\,\text{cm}$, $\rho = 10\,\text{g/cm}^3$. The drop height is then given by $h = gt^2/2$, and so we can find g from the slope of t^2 vs h. From the graph, we measure the slope $2/g = 0.127\,\text{s}^2/\text{m}$ and its error $\Delta(2/g) = 0.004\,\text{s}^2/\text{m}$ and so $g = 15.7\,\text{m/s}^2$ with an error of $\Delta g = 0.5\,\text{m/s}^2$.

$r = 50\,\text{cm}$, $\rho = 10\,\text{g/cm}^3$			
h(m)	s(m)	t(s)	t^2(s^2)
0	0.0	0.0	0.0
20	0.0	1.7	2.9
40	0.0	2.2	4.8
60	0.0	2.8	7.8
80	0.1	3.2	10.2
100	0.1	3.5	12.2
120	0.2	3.9	15.2
140	0.0	4.1	16.8
160	0.1	4.6	21.2
180	0.1	4.8	23.0
200	0.1	5.1	26.0

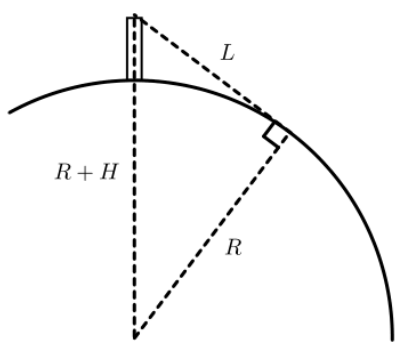


Marking scheme:		
Theory	$h = gt^2/2$	0.20 pts
Data	varying only h	0.05 pts
	maximising r	0.05 pts
	maximising ρ	0.05 pts
	table has units	0.05 pts
	h distributed roughly uniformly	0.05 pts
	$h_{\max} < 300\,\mathrm{m}$	0.05 pts
	$h_{\max} - h_{\min} > 100\,\mathrm{m}$	0.05 pts
	correct calculations of derived quantities	0.05 pts
	7 or more measurements	0.30/0.30
	6 measurements	0.25/0.30
	5 measurements	0.20/0.30
	4 or fewer measurements	0.10/0.30
Plotting	overall plot	0.30 pts
	points don't cover 60% of the area	-0.10 pts
	missing axis labels	-0.05 pts
	missing axis units	-0.05 pts
	one plotting mistake	-0.05/-0.10
	two or more plotting mistakes	-0.10/-0.10
Fit line	drawn on graph	0.05 pts
	line passes through origin	0.05 pts
	slope computed with units	0.10 pts
	uncertainty of slope computed	0.10 pts
Values	$15.0\,\mathrm{m/s^2} \leq g \leq 16.4\,\mathrm{m/s^2}$	0.20/0.20
	$14.3\,\mathrm{m/s^2} \leq g \leq 17.1\,\mathrm{m/s^2}$	0.10/0.20
	units for value	0.05 pts
	$\Delta g \leq 0.7\,\mathrm{m/s^2}$	0.20/0.20
	$\Delta g \leq 1.4\,\mathrm{m/s^2}$	0.10/0.20
	units for error	0.05 pts
	sum	**2.0 pts**

Points are added additively (including negative points), except for blocks of grey background, where the option with maximal points should be chosen (in absolute value)

A.2: How far one can see from on top of the tower can be related to the radius of the planet via the right triangle shown in the figure below. Applying the Pythagoras theorem on the triangle, one gets $(R+H)^2 = L^2 + R^2$ and so

$$R = \frac{L^2 - H^2}{2H} = 13\,200\,\mathrm{km}.$$



Marking scheme:		
Theory	correct geometry (either a figure or implicitly assumed)	0.20 pts
	correct formula	0.20 pts
Values	correct value	0.10 pts
	sum	**0.5 pts**

A.3: From Newton's law of gravity, $g = GM/R^2$. Hence,

$$M = \frac{gR^2}{G} = 4.2 \times 10^{25}\,\text{kg}.$$

By adding the errors in quadrature, we find the error

$$\Delta M = \frac{\Delta g}{g} M = 0.2 \times 10^{25}\,\text{kg}.$$

Our estimation of free-fall acceleration has a contribution from the centrifugal force caused by the rotation of the planet. This serves to reduce the acceleration on the surface and hence decrease our estimation of the planet's mass.

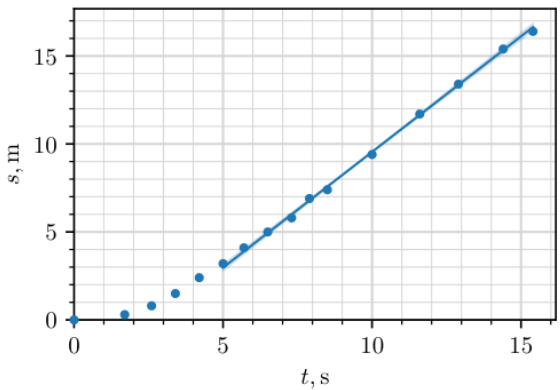
Marking scheme:		
Theory	correct formula	0.10 pts
	correct phenomena	0.20 pts
Values	$3.9 \times 10^{25}\,\text{kg} \leq M \leq 4.5 \times 10^{25}\,\text{kg}$	0.10/0.10
	$3.6 \times 10^{25}\,\text{kg} \leq M \leq 4.8 \times 10^{25}\,\text{kg}$	0.05/0.10
	missing units for value	-0.05 pts
	$\Delta M \leq 0.3 \times 10^{25}\,\text{kg}$	0.10/0.10
	$\Delta M \leq 0.6 \times 10^{25}\,\text{kg}$	0.05/0.10
	missing units for error	-0.05 pts
	sum	**0.5 pts**

The student can't get overall negative points for value nor error (for example when the value is completely out of range and the units are wrong).

B.1: In general, if the variations in gravitational acceleration are small (as is the case here as $H \ll R$), as a response to air drag, objects tend to terminal velocity where they experience no net acceleration. In the reference frame of air, this corresponds to the object falling straight down with some terminal speed v_t. In the lab frame, the object then has horizontal and vertical speeds of u and v_t respectively.

In order to find u, we can choose to drop an object that reaches terminal velocity as fast as possible and then observe how the displacement s relates to the fall time t. When terminal velocity is reached, we expect $s = s_0 + ut$, where s_0 captures the displacement related to reaching terminal velocity. To maximize the effects of air drag, we minimize radius and density, i.e. setting $\rho = 0.1\,\text{g/cm}^3$, and $r = 5\,\text{cm}$. Plotting s vs t, we measure the slope to be $u = 1.31\,\text{m/s}$ with an error of $\Delta u = 0.04\,\text{m/s}$.

$r = 5\,\text{cm},\ \rho = 0.1\,\text{g/cm}^3$		
h(m)	s(m)	t(s)
0	0.0	0.0
20	0.3	1.7
40	0.8	2.6
60	1.5	3.4
80	2.4	4.2
100	3.2	5.0
120	4.1	5.7
140	5.0	6.5
160	5.8	7.3
180	6.9	7.9
200	7.4	8.5
240	9.4	10.0
280	11.7	11.6
320	13.4	12.9
360	15.4	14.4
400	16.4	15.4



Marking scheme:		
Theory	idea of reaching terminal velocity as fast as possible	0.15 pts
	$s = s_0 + ut$	0.10 pts
Data	varying only h	0.05 pts
	minimising r	0.05 pts
	minimising ρ	0.05 pts
	table has units	0.05 pts
	h distributed roughly uniformly	0.05 pts
	$h_{\max} \geq 300\,\mathrm{m}$	0.05 pts
	$h_{\max} - h_{\min} \geq 300\,\mathrm{m}$	0.05 pts
	7 or more measurements	0.30/0.30
	6 measurements	0.25/0.30
	5 measurements	0.20/0.30
	4 or fewer measurements	0.10/0.30
Plotting	overall plot	0.30 pts
	points don't cover 60% of the area	-0.10 pts
	missing axis labels	-0.05 pts
	missing axis units	-0.05 pts
	one plotting mistake	-0.05/-0.10
	two or more plotting mistakes	-0.10/-0.10
Fit line	drawn on graph	0.10 pts
	slope computed with units	0.10 pts
	uncertainty of slope computed	0.10 pts
Values	$1.25\,\mathrm{m/s} \leq u \leq 1.37\,\mathrm{m/s}$	0.20/0.20
	$1.19\,\mathrm{m/s} \leq u \leq 1.43\,\mathrm{m/s}$	0.10/0.20
	units for value	0.05 pts
	$\Delta u \leq 0.06\,\mathrm{m/s}$	0.20/0.20
	$\Delta u \leq 0.12\,\mathrm{m/s}$	0.10/0.20
	units for error	0.05 pts
	sum	**2.0 pts**

B.2: By keeping the measurements close to the surface, we can assume to a good approximation uniform air density. Then, using similar reasoning as before, we expect $h = h_0 + v_{t0}t$, where h_0 captures the part of reaching terminal velocity.

At terminal velocity, the drag force balances out gravitational acceleration:

$$mg = 0.24A\rho_a v_t^2.$$

Using $m = 4\pi\rho r^3/3$ and $A = \pi r^2$, we get

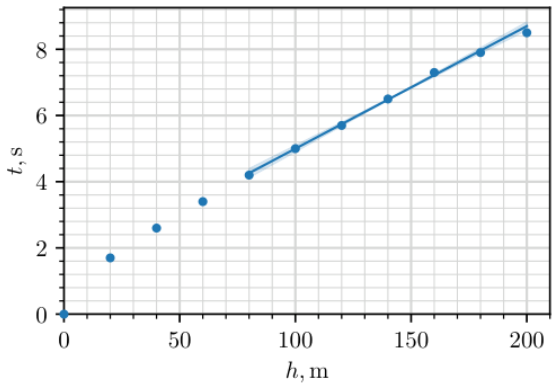
$$v_t(\rho_a) = \sqrt{\frac{4\rho r g}{3 \cdot 0.24\rho_a}}.$$

On the surface, $v_{t0} = v_t(\rho_a = \rho_{a0})$. Using the measurements from the last subtask, we can plot t vs h and measure the slope to be $1/v_{t0} = 0.037\,\mathrm{s/m}$ with an error of $\Delta(1/v_{t0}) = 0.002\,\mathrm{s/m}$. Hence, $v_{t0} = 27.0\,\mathrm{m/s}$, $\Delta v_{t0} = \Delta(1/v_{t0})/v_{t0}^2 = 2\,\mathrm{m/s}$. Now,

$$\rho_{a0} = \frac{4\rho r g}{3 \cdot 0.24 v_{t0}^2} = 0.60\,\mathrm{kg/m^3}.$$

and the error is

$$\Delta\rho_{a0} = \frac{2\Delta v_{t0}}{v_{t0}}\rho_{a0} = 0.07\,\mathrm{kg/m^3}.$$



Marking scheme:		
Theory	$h = h_0 + v_{t0}t$	0.05 pts
	formula for terminal velocity	0.10 pts
	final expression for ρ_{a0}	0.05 pts
Data	reusing the data from the last subpart	0.05 pts
	$h_{\max} \leq 200\,\mathrm{m}$	0.05 pts
	6 or more measurements	0.05 pts
Plotting	overall plot	0.25 pts
	points don't cover 60% of the area	-0.05 pts
	missing axis labels	-0.05 pts
	missing axis units	-0.05 pts
	one plotting mistake	-0.05/-0.10
	two or more plotting mistakes	-0.10/-0.10
Fit line	drawn on graph	0.05 pts
	slope computed with units	0.05 pts
	uncertainty of slope computed	0.10 pts
Values	$0.52\,\mathrm{kg/m^3} \leq \rho_{a0} \leq 0.68\,\mathrm{kg/m^3}$	0.10/0.10
	$0.44\,\mathrm{kg/m^3} \leq \rho_{a0} \leq 0.76\,\mathrm{kg/m^3}$	0.05/0.10
	$\Delta\rho_{a0} \leq 0.08\,\mathrm{kg/m^3}$	0.10/0.10
	$\Delta\rho_{a0} \leq 0.16\,\mathrm{kg/m^3}$	0.05/0.10
	units for both value and error	0.05 pts
	sum	**1.0 pts**

B.3: Due to the adiabatic profile of the atmosphere, the further up you go, the more the temperature and air density decreases, but the terminal velocity increases. We can estimate the terminal velocity of the ball at different heights by comparing the dropping time of a ball with the smallest possible terminal velocity (so minimal density and radius). This hence gives a direct probe for the air density and thus the height of the atmosphere.

If the ball reaches terminal velocity instantly, then the difference in falling time between dropping the ball at heights h_1 and $h_2 > h_1$ comes simply from $h_1 < h < h_2$. This is because in both cases the ball falls for the same amount of time at $h < h_1$ (because the terminal velocity only depends on height). Then, if $h_2 - h_1 \ll h_1$, we can estimate

$$v_t\left(\frac{h_1 + h_2}{2}\right) \approx \frac{h_2 - h_1}{t(h_2) - t(h_1)}. \tag{1}$$

In reality, the ball doesn't reach the terminal velocity instantaneously. However, it turns out we can, to a good approximation, neglect this effect. As a rough order of magnitude estimation, on the ground level, the ball experiences a time difference of $v_{t0}/(2g) = 0.8\,\mathrm{s}$ compared to the instantaneous case. This difference will increase as the ball is dropped from further up, but as long as the atmosphere isn't too much sparser in the upper parts of the tower (we can verify this later), the difference will be

insignificant compared to the total falling time of the ball. Hence, we approximate the terminal velocity via equation (1).

Because the calculated velocities are very sensitive on the measured quantities, we do repeated measurements throughout the whole height of the tower.

$r = 5\,\mathrm{cm},\ \rho = 0.1\,\mathrm{g/cm^3}$						
h(m)	s_1(m)	t_1(s)	s_2(m)	t_2(s)	s_3(m)	t_3(s)
200	7.6	8.4	7.8	8.6	7.8	8.6
400	17.0	15.7	16.9	15.6	17.3	15.7
600	26.1	22.6	25.4	22.2	26.2	22.7
800	33.6	28.5	34.6	29.2	34.3	29.1
1000	41.1	34.3	43.0	35.7	43.3	35.8
1200	51.1	41.9	50.2	41.2	50.0	41.1
1400	57.9	47.2	58.8	47.8	58.7	47.8
1600	65.5	53.0	65.1	52.8	65.3	52.9
1800	70.9	57.1	72.2	58.2	71.4	57.5
2000	78.5	62.9	79.6	63.8	79.5	63.7

Using equation (1) we make a separate table with velocities, while also adding the ground level velocity found in one of the earlier part (we set it at $h = 100\,\mathrm{m}$ because that was the centre of the range of measurements). We find air density using

$$\rho_a = \frac{4\rho r g}{3 \cdot 0.24 v_t^2}.$$

From the density profile of an adiabatic atmosphere,

$$\rho_a^{\gamma-1} = \rho_a^{0.4} = \rho_{a0}^{0.4}\left(1 - \frac{h}{H_0}\right).$$

Hence, we find H_0 by plotting $\rho_{a0}^{0.4}$ against h and fitting a straight line.

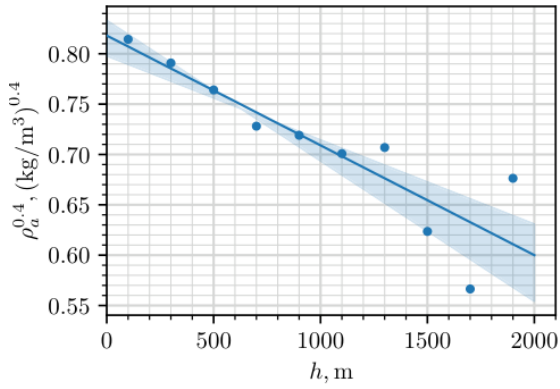
From the plot, we measure the slope $a = -\rho_{a0}^{0.4}/H_0 = -1.1 \times 10^{-4}\,(\mathrm{kg/m^3})^{0.4}/\mathrm{m}$ and the intercept $b = \rho_{a0}^{2.5} = 0.82\,(\mathrm{kg/m^3})^{0.4}$ so $H_0 = -b/a = 7500\,\mathrm{m}$. We calculate the

$r = 5\,\mathrm{cm},\ \rho = 0.1\,\mathrm{g/cm^3}$			
h(m)	v(m/s)	$\rho_a(\mathrm{kg/m^3})$	$\rho_a^{0.4}((\mathrm{kg/m^3})^{0.4})$
100	27.0	0.599	0.814
300	28.0	0.556	0.791
500	29.3	0.510	0.764
700	31.1	0.452	0.728
900	31.6	0.438	0.719
1100	32.6	0.411	0.701
1300	32.3	0.420	0.707
1500	37.7	0.307	0.624
1700	42.6	0.241	0.566
1900	34.1	0.376	0.676

error from two reasonably chosen lines that correspond to maximal and minimal estimates for H_0

$$\Delta H_0 \approx \frac{1}{2}\left(-\frac{0.80\,(\mathrm{kg/m^3})^{0.4}}{-8.4 \times 10^{-5}\,(\mathrm{kg/m^3})^{0.4}/\mathrm{m}} + \frac{0.83\,(\mathrm{kg/m^3})^{0.4}}{-1.4 \times 10^{-4}\,(\mathrm{kg/m^3})^{0.4}/\mathrm{m}}\right) \approx 2000\,\mathrm{m}.$$

We can also confirm that our assumption about the density of the atmosphere not dropping significantly in the upper parts of the tower holds true.



Alternative, less accurate solution

In this approach, it's assumed that when the air drag is maximised, the ball falls at the terminal velocity v_{t0} for the whole duration of the fall. This gives

$$\frac{\mathrm{d}h}{\mathrm{d}t} = v_t(h) = v_{t_0} \cdot \left(1 - \frac{h}{H_0}\right)^{-\frac{1}{2(\gamma-1)}}.$$

Rearranging and integrating,

$$t \approx \frac{1}{v_{t_0}} \int \mathrm{d}h \left(1 - \frac{h}{H_0}\right)^{\frac{1}{2(\gamma-1)}}.$$

So far this is exact and differs from the exact solution by the "speeding up" term which is a constant and has a smaller relative contribution the higher up one goes. In order to approximate this integral, we can do a first order binomial expansion to get

$$t \approx \frac{h}{v_{t_0}} \left(1 - \frac{1}{4H_0(\gamma-1)}h\right)$$

$$\frac{t}{h} \approx \frac{1}{v_{t_0}} - \frac{1}{4v_{t_0}H_0(\gamma-1)}h.$$

Plotting t/h vs h and calculating H_0 similarly to before (by calculating the intercept and the slope), we get $H_0 \approx 6300\,\mathrm{m}$, which falls within the error range. However, because of the approximations, this approach will be awarded a maximum of 2.0 out of 3.0 points (the following grading scheme still applies, but is capped out at 2.0).

Marking scheme:		
Theory	approximating v_{t_0} via finite difference	
	reasoning why the ball reaches terminal velocity effectively instantaneously	
	linearising v_{t_0} vs h	
	expressing H_0 in terms of the slope/intercept	
Data	varying only h	
	minimising r	
	minimising ρ	
	table has units	
	h distributed roughly uniformly	
	$h_{\max} - h_{\min} \geq 1800\,\mathrm{m}$	
	calculating derived quantities	
	15 or more measurements (can be repeat)	
	10 - 14 measurements	
	1 - 9 measurements	
Plotting	overall plot	
	points don't cover 60% of the area	
	missing axis labels	
	missing axis units	
	one plotting mistake	-
	two or more plotting mistakes	-
Fit line	drawn on graph	
	slope computed with units	
	uncertainty of slope computed	
Values	$5500\,\mathrm{m} \leq H_0 \leq 9500\,\mathrm{m}$	
	$3500\,\mathrm{m} \leq H_0 \leq 11\,500\,\mathrm{m}$	
	units for value	
	$\Delta H_0 \leq 2000\,\mathrm{m/s}$	
	$\Delta H_0 \leq 4000\,\mathrm{m/s}$	
	units for error	
	sum	

B.4: From the expression for adiabatic atmosphere we have

$$H_0 = \frac{RT_0}{\mu g}\frac{\gamma}{\gamma - 1}$$

so

$$\mu = \frac{RT_0}{H_0 g}\frac{\gamma}{\gamma - 1} = 72\,\mathrm{g\,mol^{-1}} \approx 70\,\mathrm{g\,mol^{-1}}$$

and

$$\Delta\mu = \sqrt{\frac{\Delta H_0^2}{H_0^2} + \frac{\Delta g^2}{g^2}}\,\mu = 20\,\mathrm{g\,mol^{-1}}.$$

From ideal gas law,

$$p_0 = \frac{\rho_{a0} R T_0}{\mu} = 20\,000\,\mathrm{Pa}$$

and

$$\Delta p_0 = \sqrt{\frac{\Delta\mu^2}{\mu^2} + \frac{\Delta\rho_{a0}^2}{\rho_{a0}^2}}\,p_0 = 6000\,\mathrm{Pa}.$$

Marking scheme:		
Theory	correct expression for μ	0.15 pts
	correct expression for p_0	0.15 pts
Values	$45\,\mathrm{g\,mol^{-1}} \leq \mu \leq 95\,\mathrm{g\,mol^{-1}}$	0.05 pts
	$\Delta\mu \leq 25\,\mathrm{g\,mol^{-1}}$	0.05 pts
	$12\,000\,\mathrm{Pa} \leq p_0 \leq 28\,000\,\mathrm{Pa}$	0.05 pts
	$\Delta p \leq 8000\,\mathrm{Pa}$	0.05 pts
	sum	**0.5 pts**

C.1: Our goal is to find the rotation speed Ω of the planet. The rotation of the planet affects the ball's trajectory via centrifugal and Coriolis force. The centrifugal force, however, due to $H \ll R$ is impossible to disentangle from gravitational acceleration. Coriolis force affects the ball via acceleration $\vec{a}_{\mathrm{cor}} = -2\vec{\Omega} \times \vec{v}$. This is perpendicular to both the velocity of the ball and rotation axis of the planet. Hence, it's directed along the equator, and increases linearly with the falling speed. Thus, the horizontal acceleration is given by $a_x = 2\Omega v_y + a_{\mathrm{drag}}$.

The procedure is then to minimize the effect of air drag (maximal radius and density) and hope that the Coriolis effect contributes enough to the horizontal displacement. If we neglect air drag, then $a_x = 2\Omega v_y = 2\Omega g t$ so $v_x = \int a_x \mathrm{d}t = \Omega g t^2$ and $x = \int v_x \mathrm{d}t = \Omega g t^3/3$. The final displacement will then be $s = g\Omega t_f^3/3$, where the falling time satisfies $H = g t_f^2/2$. Putting them together, we get

$$s = \frac{2\Omega}{3}\sqrt{\frac{2H^3}{g}}.$$

By varying the radius/density, we do indeed confirm that the effect of Coriolis force is significant, on the order of couple of meters. By doing a suitable number of measurements in the range 0 to 2000 m and plotting s vs $h^{1.5}$, we measure the slope

$$a = \frac{2\Omega}{3}\sqrt{\frac{2}{g}} = 5.3 \times 10^{-5}\,\mathrm{m^{-1/2}}$$

and the error

$$\Delta a = 1.1 \times 10^{-6}\,\mathrm{m^{-1/2}}$$

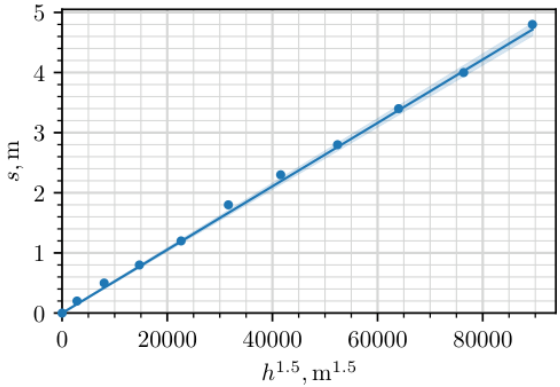
such that

$$T = \frac{2\pi}{\Omega} = \frac{4\pi}{3a}\sqrt{\frac{2}{g}} = 28\,000\,\mathrm{s} \approx 8h$$

and

$$\Delta T = \sqrt{\left(0.5\frac{\Delta g}{g}\right)^2 + \frac{\Delta a^2}{a^2}}\,T = 0.2\,\mathrm{h}.$$

$r = 50\,\mathrm{cm}$, $\rho = 10\,\mathrm{g/cm^3}$		
h(m)	s(m)	$h^{1.5}(\mathrm{m}^{1.5})$
0	0.0	0
200	0.2	2800
400	0.5	8000
600	0.8	14700
800	1.2	22600
1000	1.8	31600
1200	2.3	41600
1400	2.8	52400
1600	3.4	64000
1800	4.0	76400
2000	4.8	89400



Alternative solution.

An alternative approach is to consider the system in the non-rotating frame (where we don't have to deal with fictitious forces). In there, the ball starts off with speed $v_0 = \Omega(R + H)$. Due to the conservation of angular momentum, as the ball drops towards the ground, the ball's angular speed will start increasing and the ground will start lagging behind (the ground rotates with Ω). At height h, when the ball moves with angular speed ω, the conservation of angular momentum reads $\omega(R+h)^2 = \Omega(R+H)^2$ and so the angular lag between the ball and the ground is

$$\Delta\omega = \omega - \Omega = \Omega\left(\left(\frac{R+H}{R+h}\right)^2 - 1\right) \approx 2\Omega\frac{H-h}{R}.$$

The positional velocity shift along the ground is then $v_x = \Delta\omega R = 2\Omega(H-h) = \Omega g t^2$. We recover the same expression as for Coriolis force, and from there we proceed the same way as before.

Marking scheme:		
Theory	Deriving $s(h)$	0.80 pts
	linearising s vs h	0.10 pts
Data	varying only h	0.05 pts
	minimising r and ρ	0.05 pts
	table has units	0.05 pts
	h distributed roughly uniformly	0.05 pts
	$h_{\max} - h_{\min} \geq 1800\,\mathrm{m}$	0.05 pts
	calculating derived quantities	0.05 pts
	7 or more measurements	0.30/0.30
	6 measurements	0.25/0.30
	5 measurements	0.20/0.30
	4 or fewer measurements	0.10/0.30
Plotting	overall plot	0.30 pts
	points don't cover 60% of the area	-0.10 pts
	missing axis labels	-0.05 pts
	missing axis units	-0.05 pts
	one plotting mistake	-0.05/-0.10
	two or more plotting mistakes	-0.10/-0.10
Fit line	drawn on graph	0.10 pts
	slope computed with units	0.10 pts
	uncertainty of slope computed	0.10 pts
Values	$27\,000\,\mathrm{s} \leq T \leq 29\,000\,\mathrm{s}$	0.20/0.20
	$26\,000\,\mathrm{s} \leq T \leq 30\,000\,\mathrm{s}$	0.10/0.20
	missing units for value	-0.05 pts
	$\Delta T \leq 1000\,\mathrm{s}$	0.20/0.20
	$\Delta T \leq 2000\,\mathrm{s}$	0.10/0.20
	missing units for error	-0.05 pts
	sum	**2.5 pts**