A. Earth as a blackbody

A-1. All the energy emitted from the surface of the Sun, will reach a sphere of radius d , therefore:

$$\sigma T_{\mathrm{S}}^4.(4\pi R_{\mathrm{S}}^2) = (4\pi d^2).S_0$$

$$S_0 = \sigma T_{\mathrm{S}}^4.\left(\frac{R_{\mathrm{S}}}{d}\right)^2 = 5.67{\times}10^{-8}\,\frac{\mathrm{W}}{\mathrm{m^2K^4}}{\times}(5.77{\times}10^3\,\mathrm{K})^4{\times}\left(\frac{6.96\times10^8\,\mathrm{m}}{1.5\times10^{11}\,\mathrm{m}}\right)^2 = 1.35{\times}10^3\,\frac{\mathrm{W}}{\mathrm{m^2}}$$

A-1 (0.6 pt)

$$S_0 = \sigma T_{\mathrm{S}}^4.\left(\frac{R_{\mathrm{S}}}{d}\right)^2$$

, Numerical value of $S_0 = 1.35\times10^3$ W/m^2

A-2. It is assumed that the Earth is in thermal equilibrium. Therefore, the energy it receives per unit time should be equal to the energy it radiates per unit time. The Earth's cross-section intercepting the solar radiation at this distance has an area of πR_{E}^2 , but the Earth radiates heat from all points on its surface with an area of $4\pi R_{\mathrm{E}}^2$, so:

$$\pi R_{\mathrm{E}}^2.S_0 = 4\pi R_{\mathrm{E}}^2\sigma T_{\mathrm{E}}^4 \rightarrow T_{\mathrm{E}} = \left(\frac{S_0}{4\sigma}\right)^{\frac{1}{4}} = 278\,\mathrm{K}$$

A-2 (0.6 pt)

$$T_{\mathrm{E}} = \left(\frac{S_0}{4\sigma}\right)^{\frac{1}{4}} = \sqrt{\frac{R_{\mathrm{S}}}{2d}}\,T_{\mathrm{S}}$$

, Numerical value of $T_{\mathrm{E}} = 278$ K

A-3. The radiation is maximum at the wavelength for which the derivative of u with respect to λ is zero:

$$\frac{du}{d\lambda} = \frac{2\pi hc^2}{\lambda^6}.\frac{1}{\exp\left(\frac{hc}{\lambda k_{\mathrm{B}}T}\right)-1}.\left[-5+\frac{hc}{\lambda k_{\mathrm{B}}T}\,\frac{\exp\left(\frac{hc}{\lambda k_{\mathrm{B}}T}\right)}{\exp\left(\frac{hc}{\lambda k_{\mathrm{B}}T}\right)-1}\right]$$

$$\left.\frac{du}{d\lambda}\right|_{\lambda=\lambda_{\mathrm{m}}} = 0 \quad\Rightarrow\quad \frac{\frac{hc}{\lambda_{\mathrm{m}}k_BT}\,\exp\left(\frac{hc}{\lambda_{\mathrm{m}}k_{\mathrm{B}}T}\right)}{\exp\left(\frac{hc}{\lambda_{\mathrm{m}}K_{\mathrm{B}}T}\right)-1} = 5$$

Defining $x_{\mathrm{m}} \equiv \frac{hc}{\lambda_{\mathrm{m}}k_{\mathrm{B}}T}$ we obtain the following transcendental equation:

$$5\left(1-e^{-x_{\mathrm{m}}}\right)-x_{\mathrm{m}} = 0$$

A-3 (0.4 pt)

$$f(x) = 5\left(1-e^{-x}\right)-x$$

A-4. The first guess is $x_{\mathrm{m}}^{(1)} = 5$. Substituting repeatedly for x_{m} we can continue as follows:

$$x_{\mathrm{m}}^{(2)} = 5\left(1-e^{-5}\right) = 4.97$$

$$x_{\mathrm{m}}^{(3)} = 5\left(1-e^{-4.97}\right) = 4.97$$

Further iterations do not change the value of x_{m} to three significant figures, so:

$$\lambda_{\mathrm{m}}T = \frac{hc}{x_{\mathrm{m}}k_{\mathrm{B}}} = b = 1240\,\mathrm{eV\cdot nm}\times\frac{1}{4.97\times8.62\times10^{-5}\,\mathrm{eVK^{-1}}} = 2.89\times10^6\,\mathrm{nm\cdot K}$$

A-4 (0.4 pt)

$x_{\mathrm{m}} = \{4.96, 4.97\}$, Numerical value of $b = [2.89, 2.90] \times 10^6$ nm·K

A-5. Using Wien's displacement law and the constant b obtained in the previous part, we can calculate the wavelength at which the radiation from the Sun and the Earth reaches its maximum:

$$\lambda_{\max}^{\mathrm{Sun}} = \frac{b}{T_{\mathrm{S}}} = \frac{2.89 \times 10^6\,\mathrm{nm \cdot K}}{5.77 \times 10^3\,\mathrm{K}} = [5.01, 5.02] \times 10^2\,\mathrm{nm}$$

$$\lambda_{\max}^{\mathrm{Earth}} = \frac{b}{T_{\mathrm{E}}} = \frac{2.89 \times 10^6\,\mathrm{nm \cdot K}}{278\,\mathrm{K}} = 1.04 \times 10^4\,\mathrm{nm}$$

A-5 (0.2 pt)

$\lambda_{\max}^{\mathrm{Sun}} = [5.01, 5.02] \times 10^2$ nm , $\lambda_{\max}^{\mathrm{Earth}} = 1.04 \times 10^4$ nm

A-6. From the diagram, it can clearly be seen that $\gamma \tilde{u}_{\mathrm{S}}(\lambda_{\max}^{\mathrm{S}}) = u(\lambda_{\max}^{\mathrm{Earth}}, T_{\mathrm{E}})$, so we have:

$$\tilde{u}_{\mathrm{S}}(\lambda_{\max}^{\mathrm{Sun}}) = \left(\frac{R_{\mathrm{S}}}{d}\right)^2 \frac{2\pi hc^2}{(\lambda_{\max}^{\mathrm{Sun}})^5} \frac{1}{\exp\left(\frac{hc}{\lambda_{\max}^{\mathrm{Sun}} k_{\mathrm{B}} T_{\mathrm{S}}}\right) - 1} = \left(\frac{R_{\mathrm{S}}}{d}\right)^2 \frac{2\pi hc^2}{(\lambda_{\max}^{\mathrm{Sun}})^5} \frac{1}{\exp\left(\frac{hc}{k_{\mathrm{B}} b}\right) - 1}$$

$$u\left(\lambda_{\max}^{\mathrm{Earth}}, T_{\mathrm{E}}\right) = \frac{2\pi hc^2}{(\lambda_{\max}^{\mathrm{Earth}})^5} \frac{1}{\exp\left(\frac{hc}{\lambda_{\max}^{\mathrm{Earth}} k_{\mathrm{B}} T_{\mathrm{E}}}\right) - 1} = \frac{2\pi hc^2}{(\lambda_{\max}^{\mathrm{Earth}})^5} \frac{1}{\exp\left(\frac{hc}{k_{\mathrm{B}} b}\right) - 1}$$

Dividing these two quantities we'll find:

$$\gamma = \left(\frac{d}{R_{\mathrm{S}}}\right)^2 \times \left(\frac{T_{\mathrm{E}}}{T_{\mathrm{S}}}\right)^5 = [1.20, 1.21] \times 10^{-2}$$

A-6 (0.8 pt)

$$\gamma = \left(\frac{d}{R_{\mathrm{S}}}\right)^2 \times \left(\frac{T_{\mathrm{E}}}{T_{\mathrm{S}}}\right)^5 = \left(\frac{d}{R_{\mathrm{S}}}\right)^2 \times \left(\frac{\lambda_{\max}^{\mathrm{Sun}}}{\lambda_{\max}^{\mathrm{Earth}}}\right)^5$$

, Numerical value of $\gamma = [1.20, 1.21] \times 10^{-2}$

B. The Greenhouse Effect

B-1. Both the Earth and its atmosphere are in thermal equilibrium, so one can write an equation that balances the input and output powers. For the Earth we have:

$$\left(\pi R_{\mathrm{E}}^2\right)(1 - r_{\mathrm{A}})\, S_0 + \left(4\pi R_{\mathrm{E}}^2\right) \sigma T_{\mathrm{A}}^4 = \left(4\pi R_{\mathrm{E}}^2\right) \sigma T_{\mathrm{E}}^4,$$

and for the atmosphere:

$$\left(4\pi R_{\mathrm{E}}^2\right) \sigma T_{\mathrm{E}}^4 = 2\left(4\pi R_{\mathrm{E}}^2\right) \sigma T_{\mathrm{A}}^4.$$

Note that the coefficient 2 on the right-hand side of the equation is due to the atmosphere radiating heat on both sides (above and below). Eliminating T_E from the two relations we obtain:

$$T_{\mathrm{A}} = \left(\frac{(1 - r_{\mathrm{A}})\frac{S_0}{4}}{\sigma}\right)^{\frac{1}{4}} = 2.58 \times 10^2\,\mathrm{K} \quad \Rightarrow \quad T_{\mathrm{E}} = \left(2T_{\mathrm{A}}^4\right)^{\frac{1}{4}} = 3.07 \times 10^2\,\mathrm{K}$$

B-1 (1.0 pt)

$$T_A = \left(\frac{(1 - r_{\mathrm{A}})\frac{S_0}{4}}{\sigma}\right)^{\frac{1}{4}}$$

, Numerical value of $T_A = 2.58 \times 10^2$ K

$$T_{\rm E} = \left(\frac{(1-r_{\rm A})\frac{S_0}{2}}{\sigma} \right)^{\frac{1}{4}}$$

, Numerical value of $T_{\rm E} = 3.07 \times 10^2$ K

B-2. As can be seen in the figure, a fraction $(1 - r_{\rm A})$ of the solar radiation reaches the Earth's surface after traversing the atmosphere. A fraction $r_{\rm E}$ of this light is reflected back and reaches the atmosphere, where a fraction $r_{\rm A}$ is reflected and returns to the Earth's surface. This process repeats *ad infinitum* and the sum of the powers transmitted at all these instances, determines the albedo. Denoting the power returned to space after n reflections by $\tilde{S}_n$, we'll have $\tilde{S}_0 = r_{\rm A} S_0$ and the remaining power i.e. $(1 - r_{\rm A})S_0$, reaches the Earth's surface. From this power, $(1 - r_{\rm A})r_{\rm E}S_0$ is reflected, and a fraction $1 - r_{\rm A}$ of it is transmitted through the atmosphere to the space, hence:

$$\tilde{S}_1 = (1 - r_{\rm A})^2 r_{\rm E} S_0 = \frac{(1 - r_{\rm A})^2}{r_{\rm A}} \, r_{\rm E} \tilde{S}_0$$

The power that is reflected back to the Earth by the atmosphere after $(n-1)$ reflections is $\tilde{S}_{n-1} \left(\frac{r_{\rm A}}{1-r_{\rm A}} \right)$, of which a fraction $r_{\rm E}$ is again sent back towards the atmosphere on the n'th reflection, and the atmosphere allows a fraction $1 - r_{\rm A}$ of this reflected power to escape into the space, thus:

$$\tilde{S}_n = \frac{\tilde{S}_{n-1}}{1 - r_{\rm A}} \, r_{\rm A} \, r_{\rm E} \times (1 - r_{\rm A}) = r_{\rm A} r_{\rm E} \tilde{S}_{n-1} = (r_{\rm A} r_{\rm E})^{n-1} \, \tilde{S}_1$$

By adding all these terms, one obtains the power returned per unit area from the Earth-atmosphere system:

$$\begin{aligned} \tilde{S} &= \sum_{n=0}^{\infty} \tilde{S}_n = \tilde{S}_0 + \tilde{S}_1 \sum_{n=1}^{\infty} (r_{\rm A} r_{\rm E})^{n-1} = r_{\rm A} S_0 + (1 - r_{\rm A})^2 r_{\rm E} S_0 \times \frac{1}{1 - r_{\rm A} r_{\rm E}} \\ &= \left[r_{\rm A} + \frac{(1 - r_{\rm A})^2 r_{\rm E}}{1 - r_{\rm A} r_{\rm E}} \right] \times S_0 \end{aligned}$$

Dividing by the solar constant we get the value for albedo:

$$\alpha = \frac{\tilde{S}}{S_0} = r_{\rm A} + \frac{(1 - r_{\rm A})^2 r_{\rm E}}{1 - r_{\rm A} r_{\rm E}} = 3.13 \times 10^{-1}$$

B-2 (1.6 pt)

$$\alpha = r_{\rm A} + \frac{(1 - r_{\rm A})^2 r_{\rm E}}{1 - r_{\rm A} r_{\rm E}}$$

, Numerical value of $\alpha = 3.13 \times 10^{-1}$

B-3. Again, thermal equilibrium requires the input and output powers to be equal both for the Earth and for the atmosphere, the only difference being that the Earth absorbs now a fraction $1 - \alpha$ of the Sun's radiation. Thus, for Earth we have:

$$\left(4\pi R_{\rm E}^2\right) \epsilon \sigma T_{\rm A}^4 + \left(\pi R_{\rm E}^2\right) (1 - \alpha) S_0 = \left(4\pi R_{\rm E}^2\right) \sigma T_{\rm E}^4,$$

and for the atmosphere:

$$\left(4\pi R_{\rm E}^2\right) \epsilon \sigma T_{\rm E}^4 = 2 \left(4\pi R_{\rm E}^2\right) \epsilon \sigma T_{\rm A}^4$$

$$T_{\rm E} = \left[\frac{(1 - \alpha)}{2\sigma(2 - \epsilon)} S_0 \right]^{\frac{1}{4}}, \qquad T_{\rm A} = \left(\frac{T_{\rm E}^4}{2} \right)^{\frac{1}{4}}$$

$$\epsilon = \frac{\left[\sigma T_{\rm E}^4 - \frac{(1-\alpha)}{4} S_0 \right]}{\sigma T_{\rm A}^4} = 2 \, \frac{\left[\sigma T_{\rm E}^4 - \frac{(1-\alpha)}{4} S_0 \right]}{\sigma T_{\rm E}^4} = [8.07, 8.11] \times 10^{-1}$$

B-3 (1.0 pt)

$$T_{\rm E} = \left[\frac{(1-\alpha)}{2\sigma(2-\epsilon)} S_0\right]^{\frac{1}{4}}$$

, Numerical value of $\epsilon = [8.07, 8.11] \times 10^{-1}$

B-4.

$$\frac{dT_{\rm E}}{d\epsilon} = \frac{1}{4}\left[\frac{(1-\alpha)S_0}{2\sigma(2-\epsilon)}\right]^{\frac{1}{4}} \frac{1}{(2-\epsilon)}$$

$$dT_{\rm E} = \frac{dT_{\rm E}}{d\epsilon}\,\epsilon\,\frac{d\epsilon}{\epsilon} = \left[\frac{4\sigma T_{\rm E}^4}{(1-\alpha)S_0} - 1\right]\frac{T_{\rm E}}{4} \times 0.01 = [4.87, 4.92] \times 10^{-1}$$

B-4 (0.8pt)

$$\frac{dT_{\rm E}}{d\epsilon} = \frac{1}{4}\left[\frac{(1-\alpha)S_0}{2\sigma(2-\epsilon)}\right]^{\frac{1}{4}} \frac{1}{(2-\epsilon)}$$

, Numerical value of $\delta T_{\rm E} = [4.87, 4.92] \times 10^{-1}$ K

B-5. The equations for thermal equilibrium are similar to those for Part B.3, only a non-radiative thermal current needs to be added. For the Earth:

$$\left(\pi R_{\rm E}^2\right)(1-\alpha)S_0 + \left(4\pi R_{\rm E}^2\right)\epsilon\sigma T_{\rm A}^4 = \left(4\pi R_{\rm E}^2\right)\sigma T_{\rm E}^4 + \left(4\pi R_{\rm E}^2\right)k\left(T_{\rm E} - T_{\rm A}\right),$$

and for the atmosphere:

$$\left(4\pi R_{\rm E}^2\right)\epsilon\sigma T_{\rm E}^4 + \left(4\pi R_{\rm E}^2\right)k\left(T_{\rm E} - T_{\rm A}\right) = 2\left(4\pi R_{\rm E}^2\right)\epsilon\sigma T_{\rm A}^4.$$

After completing the calculations, we will have:

$$\epsilon = \frac{\sigma T_{\rm E}^4 - (1-\alpha)\frac{S_0}{4}}{\sigma\left(T_{\rm E}^4 - T_{\rm A}^4\right)} = [8.47, 8.52] \times 10^{-1}$$

$$k = \frac{\epsilon\sigma\left(2T_{\rm A}^4 - T_{\rm E}^4\right)}{T_{\rm E} - T_{\rm A}} = \frac{\left(2T_{\rm A}^4 - T_{\rm E}^4\right) \times \left[\sigma T_{\rm E}^4 - (1-\alpha)\frac{S_0}{4}\right]}{\left(T_{\rm E}^4 - T_{\rm A}^4\right) \times \left(T_{\rm E} - T_{\rm A}\right)} = [3.57, 3.66] \times 10^{-1}\,{\rm W/m^2K}$$

B-5 (1.6pt)

$$\epsilon = \frac{\sigma T_{\rm E}^4 - (1-\alpha)\frac{S_0}{4}}{\sigma\left(T_{\rm E}^4 - T_{\rm A}^4\right)}$$

, Numerical value of $\epsilon = [8.47, 8.52] \times 10^{-1}$

$$k = \frac{\left(2T_{\rm A}^4 - T_{\rm E}^4\right) \times \left[\sigma T_{\rm E}^4 - (1-\alpha)\frac{S_0}{4}\right]}{\left(T_{\rm E}^4 - T_{\rm A}^4\right) \times \left(T_{\rm E} - T_{\rm A}\right)}$$

, Numerical value of $k = [3.57, 3.66] \times 10^{-1}$ W/m^{2}K

B-6. In order to find the change in the temperatures of the Earth and the atmosphere in terms of ϵ and k, we take the logarithm of both sides of the relations before taking the derivative:

$$\ln\epsilon = \ln\left[\sigma T_{\rm E}^4 - (1-\alpha)\frac{S_0}{4}\right] - \ln\sigma - \ln\left(T_{\rm E}^4 - T_{\rm A}^4\right)$$

$$\ln k = \ln\epsilon + \ln\sigma + \ln\left(2T_{\rm A}^4 - T_{\rm E}^4\right) - \ln\left(T_{\rm E} - T_{\rm A}\right)$$

$$\frac{1}{\epsilon} = \frac{4\sigma T_{\rm E}^3 \frac{dT_{\rm E}}{d\epsilon}}{\sigma T_{\rm E}^4 - (1-\alpha)\frac{S_0}{4}} - \frac{4T_{\rm E}^3\frac{dT_{\rm E}}{d\epsilon} - 4T_{\rm A}^3\frac{dT_{\rm A}}{d\epsilon}}{T_{\rm E}^4 - T_{\rm A}^4}$$

$$0 = \frac{1}{\epsilon} + \frac{8T_{\rm A}^3\frac{dT_{\rm A}}{d\epsilon} - 4T_{\rm E}^3\frac{dT_{\rm E}}{d\epsilon}}{2T_{\rm A}^4 - T_{\rm E}^4} - \frac{\frac{dT_{\rm E}}{d\epsilon} - \frac{dT_{\rm A}}{d\epsilon}}{T_{\rm E} - T_{\rm A}}$$

$$\epsilon\left[\frac{1}{T_{\rm E}-T_{\rm A}}+\frac{4T_{\rm E}^3}{2T_{\rm A}^4-T_{\rm E}^4}\right]\frac{dT_{\rm E}}{d\epsilon}=1+\epsilon\left[\frac{8T_{\rm A}^3}{2T_{\rm A}^4-T_{\rm E}^4}+\frac{1}{T_{\rm E}-T_{\rm A}}\right]\frac{dT_{\rm A}}{d\epsilon}$$

$$1+\epsilon\left[\frac{4T_{\rm E}^3}{T_{\rm E}^4-T_{\rm A}^4}-\frac{4\sigma T_{\rm E}^3}{\sigma T_{\rm E}^4-(1-\alpha)\frac{S_0}{4}}\right]\frac{dT_{\rm E}}{d\epsilon}=\frac{4T_{\rm A}^3}{T_{\rm E}^4-T_{\rm A}^4}\,\epsilon\,\frac{dT_{\rm A}}{d\epsilon}$$

Solving this set of linear equations and substituting ϵ in B-5, we find:

$$\frac{dT_{\rm E}}{d\epsilon}=\frac{\left[\frac{\sigma(T_{\rm E}^4-T_{\rm A}^4)}{\sigma T_{\rm E}^4-(1-\alpha)\frac{S_0}{4}}\right]\left[1+\left(\frac{T_{\rm E}^4-T_{\rm A}^4}{4T_{\rm A}^3}\right)\left[\frac{8T_{\rm A}^3}{2T_{\rm A}^4-T_{\rm E}^4}+\frac{1}{T_{\rm E}-T_{\rm A}}\right]\right]}{\left[\frac{1}{T_{\rm E}-T_{\rm A}}+\frac{4T_{\rm E}^3}{2T_{\rm A}^4-T_{\rm E}^4}\right]-\left(\frac{\sigma T_{\rm A}^4-(1-\alpha)\frac{S_0}{4}}{\sigma T_{\rm E}^4-(1-\alpha)\frac{S_0}{4}}\right)\left(\frac{T_{\rm E}}{T_{\rm A}}\right)^3\left[\frac{8T_{\rm A}^3}{2T_{\rm A}^4-T_{\rm E}^4}+\frac{1}{T_{\rm E}-T_{\rm A}}\right]}$$

$$\epsilon\frac{dT_{\rm E}}{d\epsilon}=\frac{1+\left(\frac{T_{\rm E}^4-T_{\rm A}^4}{4T_{\rm A}^3}\right)\left[\frac{8T_{\rm A}^3}{2T_{\rm A}^4-T_{\rm E}^4}+\frac{1}{T_{\rm E}-T_{\rm A}}\right]}{\left[\frac{1}{T_{\rm E}-T_{\rm A}}+\frac{4T_{\rm E}^3}{2T_{\rm A}^4-T_{\rm E}^4}\right]-\left(\frac{\sigma T_{\rm A}^4-(1-\alpha)\frac{S_0}{4}}{\sigma T_{\rm E}^4-(1-\alpha)\frac{S_0}{4}}\right)\left(\frac{T_{\rm E}}{T_{\rm A}}\right)^3\left[\frac{8T_{\rm A}^3}{2T_{\rm A}^4-T_{\rm E}^4}+\frac{1}{T_{\rm E}-T_{\rm A}}\right]}$$

$$dT_{\rm E}=\epsilon\frac{dT_{\rm E}}{d\epsilon}\frac{d\epsilon}{\epsilon}=[5.21,5.28]\times10^{-1}\,{\rm K}$$

B-6 (1.0pt)

(a)

$$\begin{cases}\epsilon\left[\frac{1}{T_{\rm E}-T_{\rm A}}+\frac{4T_{\rm E}^3}{2T_{\rm A}^4-T_{\rm E}^4}\right]\frac{dT_{\rm E}}{d\epsilon}=1+\epsilon\left[\frac{8T_{\rm A}^3}{2T_{\rm A}^4-T_{\rm E}^4}+\frac{1}{T_{\rm E}-T_{\rm A}}\right]\frac{dT_{\rm A}}{d\epsilon}\\ 1+\epsilon\left[\frac{4T_{\rm E}^3}{T_{\rm E}^4-T_{\rm A}^4}-\frac{4\sigma T_{\rm E}^3}{\sigma T_{\rm E}^4-(1-\alpha)\frac{S_0}{4}}\right]\frac{dT_{\rm E}}{d\epsilon}=\frac{4T_{\rm A}^3}{T_{\rm E}^4-T_{\rm A}^4}\,\epsilon\,\frac{dT_{\rm A}}{d\epsilon}\end{cases}$$

(b) $\delta T_{\rm E}=[5.21,5.28]\times10^{-1}$ K