QUESTION T3 [10 pts]. Chasing the absolute zero

A refrigeration cycle must be carried out by a system with characteristics appropriate for the temperatures to be achieved. One hundred years ago, in 1926, the chemist William Giauque and, independently, the physicist Peter Debye proposed paramagnetic salts as ideal systems for lowering temperatures far below 1 K. Indeed, beginning in the following decade, temperatures as low as 0.0044 K were achieved (Wander de Haas and Gerrit Wiersma, 1935, Leiden Cryogenic Laboratories). In this question, the paramagnetic Carnot refrigeration cycle is studied.

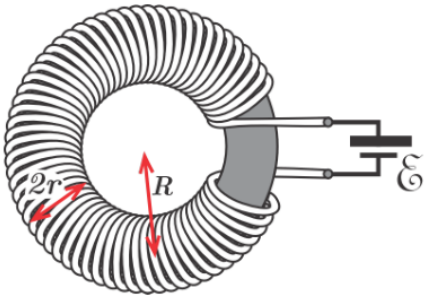

Fig. 3a.

Consider a torus with mean radius R and inner radius r made of a homogeneous, isotropic paramagnetic material, around which an insulated conducting wire is wound, with its ends connected to an external voltage source (emf), as shown in figure 3a. The resistance of the wire is so low that energy losses due to the heating of the wire can be neglected. The winding is dense, with N turns in total, and it is assumed that $r \ll R$, so that the fields $\vec{H}$ and $\vec{B}$ and the magnetization $\vec{M}$ have approximately constant magnitudes throughout the torus. Let V and A be the volume and the cross-section area of the torus. Hereinafter, the abbreviation Pm-T is used to refer to the paramagnetic torus.

In paramagnetic materials, $\vec{M}$ is parallel to $\vec{H}$ and the following relation holds:

$$\vec{B} = \mu_0 \vec{H} + \mu_0 \vec{M}.$$

Energy transfer sign convention: in this problem, Work (W) and Heat (Q) are considered positive when they flow into the system, and negative when they flow out.

T3-A [1.0 pts]. Work in the Pm-T

Hint: in magnetic materials, $\oint_C \vec{H} \cdot d\vec{\ell} = I_C$, where I_C is the net free current passing through the area bounded by the closed curve C.

T3-A1 [0.2 pts]. Let H be the magnitude of $\vec{H}$ in the torus. Write H in terms of N, A, V and the instantaneous electric current I in the wire.

T3-A2 [0.6 pts]. Let dW_{emf} be the work that the external voltage source performs to change the magnitude of $\vec{B}$ by dB. Write dW_{emf} in terms of V, H, and dB.

T3-A3 [0.2 pts]. The total work done by the voltage source, dW_{emf}, can be divided in two parts: the work that would be performed to change the magnetic field if the toroid had a vacuum core, dW_{vac}, and the work done on the paramagnetic material itself, dW. Write dW in terms of μ_0, H, V and dM.

T3-B [3.0 pts]. Heat and temperature in the Pm-T

The following table compares two thermodynamic systems: the ideal gas and the Pm-T. Use this information and your answer to question T3-A3 to solve the following questions. This information will also be used in part C. If you did not find an answer for dW in question T3-A3, you may use $dW = \alpha V \mu_0 H dM$ where α is a number. The volume V of the paramagnetic torus is constant, so that there is no work due to pressure. Below, T is the temperature in all instances.

IDEAL GAS	PARAMAGNETIC TORUS
State functions: P, V and temperature T	State functions: H, M and temperature T
Constants: R, γ and number of moles n	Constants: K, λ and number of moles n
Equation of state: $PV = nRT$	Equation of state: $TMV = nKH$
Heat capacity at constant V: $C_V = nR/(\gamma - 1)$	Heat capacity at constant M: $C_M = n\lambda/T^2$
Internal energy U in any process satisfies $dU = C_V dT$	Internal energy U in any process satisfies $dU = C_M dT$

K and λ are constants given by the material in the torus. You may use the equations provided in the table directly without proving them.

T3-B1 [1.5 pts]. If the magnitude of $\vec{H}$ changes from H_i to H_f at a constant temperature T, an energy Q is transferred between the Pm-T and its environment. Write Q in terms of μ_0, n, V, K, H_i, H_f, and T.

T3-B2 [1.5 pts]. If the magnitude of $\vec{H}$ changes adiabatically from H_i to H_f, the temperature of the Pm-T changes from T_i to T_f. Write $\Delta T = T_f - T_i$ in terms of μ_0, V, n, K, λ, H_i, H_f, and T_i.

T3-C [6.0 pts]. Carnot refrigerator of the Pm-T

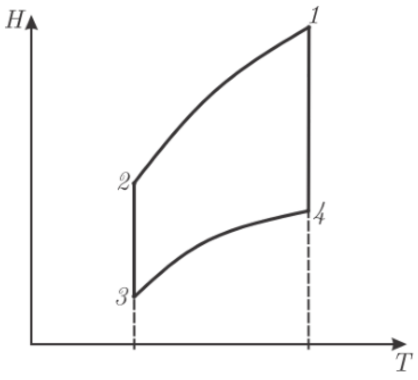

Fig.3b.

The H vs. T diagram of the Carnot refrigeration cycle $1 \to 2 \to 3 \to 4 \to 1$ performed by the Pm-T is shown in figure 3b. In this cycle, let:

T_h: temperature of the hot reservoir
T_c: temperature of the cold reservoir
Q_h: absolute value of the heat transferred to the hot reservoir
Q_c: absolute value of the heat absorbed from the cold reservoir

T3-C1 [0.2 pts]. Mark T_h and T_c on the T axis of the H vs. T diagram. Also label the processes in which the transfers of Q_h and Q_c occur, respectively.

T3-C2 [1.5 pts]. Let M_1, M_2, M_3, and M_4 be the magnitudes of $\vec{M}$ at the vertices of this cycle. Write M_1 in terms of M_2, M_3, and M_4.

T3-C3 [0.8 pts]. Suppose that to cool 1.00 L of liquid helium initially at 1.00 K, we utilize a 2.0 moles Pm-T made of *potassium chromate*. For this material, $K = 1.87 \times 10^{-6}\,\mathrm{K \cdot m^3/mol}$, its density is $2730\,\mathrm{kg/m^3}$ and its molar mass is $0.19\,\mathrm{kg/mol}$. The Pm-T performs a Carnot refrigeration cycle with $H_1 = 411624\,\mathrm{A/m}$, $H_2 = 311306\,\mathrm{A/m}$, $H_3 = 204618\,\mathrm{A/m}$ and $H_4 = 240446\,\mathrm{A/m}$. After one operating cycle, calculate the final temperature of the liquid helium. Assume that for the liquid helium both the specific heat capacity $c = 100\,\mathrm{J/(kg \cdot K)}$ and the density $\rho = 130\,\mathrm{kg/m^3}$ remain constant.

T3-C4 [2.0 pts]. In order to continuously lower the temperature of a body with constant heat capacity C_c, Carnot cycles are allowed to occur over time in such a way that the power P transferred to the refrigerator remains constant. It is then assumed that the temperature T_h of the hot reservoir remains constant and that in each operating cycle the temperature T_c of the cooled body decreases by dT_c, satisfying the relation $dQ_c/dQ_h = T_c/T_h$. If at $t = 0$ we have $T_c = T_0$, determine the time t that the Carnot refrigerator must operate for the cooled body to reach a temperature $T < T_0$. Express your answer in terms of C_c, T_h, P, T_0 and T.

T3-C5 [1.5 pts]. The efficiency of a refrigerator is quantified by its *coefficient of performance* $\mathrm{COP} = Q_c/W$. Determine the overall COP of the refrigerator of question T3-C4 with all the cycles performed until time t. Write your answer in terms of T_0 and T_h and T.