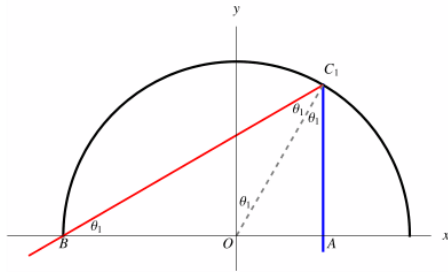


SOLUTION T2 [10 pts]. Cooking with sunlight?

PART A [1.5 pts]. Multiple reflections

T2-A1 [1.5 pts]. For $n = 1$, finding x_1 corresponds to considering the limiting case of the ray, shown in blue in the figure below.



In this case, the incident ray passes through point A , and the reflected ray, shown in red in the figure, passes exactly through point B on the edge of the mirror. Let θ_1 be the angle formed by the radius passing through the reflection point C_1 with the vertical axis (the y axis). Let O be the center of the mirror. Since the incident ray is parallel to the y axis, the angle of incidence is

$$\angle OC_1A = \theta_1 .$$

By the law of reflection,

$$\angle OC_1A = \angle OC_1B = \theta_1 .$$

Additionally, note that the triangle C_1BO is isosceles, so

$$\angle C_1BO = \theta_1 .$$

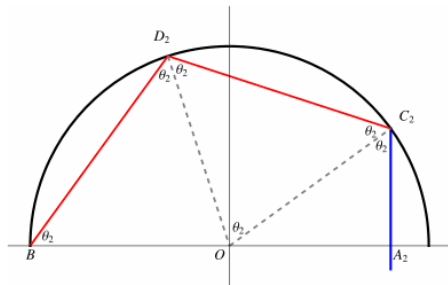
Finally, summing the internal angles of the triangle C_1BA , we have

$$\begin{aligned} 3\theta_1 + \frac{\pi}{2} &= \pi, \\ \theta_1 &= \frac{\pi}{6}. \end{aligned}$$

Correspondingly, from the triangle OC_1A , we obtain

$$x_1 = R \sin \frac{\pi}{6} = \frac{R}{2} .$$

For $n = 2$, the limiting case for two reflections is shown in the figure below.



In analogy with the case $n = 1$, we let θ_2 be the angle between the radius passing through the reflection point C_2 and the mirror axis. We see that the trajectory of the reflected ray determines two isosceles triangles with a common side and a base angle of θ_2 . Also note that the ray's trajectory forms half of a regular pentagon. Summing over the internal angles of the two isosceles triangles and the rectangular triangle, we obtain

$$5\theta_2 + \frac{\pi}{2} + \pi = 3\pi,$$

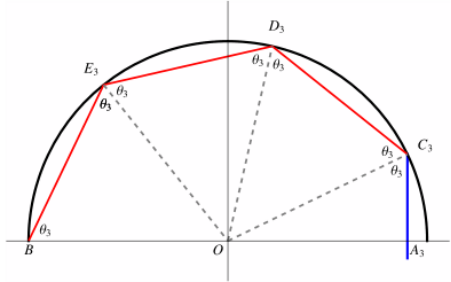
which simplifies to

$$\theta_2 = \frac{3\pi}{10}.$$

Correspondingly,

$$x_2 = R \sin \frac{3\pi}{10} = \frac{1 + \sqrt{5}}{4} R \approx 0.8R.$$

Case n=3: For completeness, we present the limiting case for 3 reflections in the figure below.



Letting θ_3 be the incidence angle, we observe that the trajectory of the reflected ray determines three isosceles triangles with common sides and a base angle of θ_3 . Also notice that the ray's trajectory now forms half of a regular heptagon. Summing over the internal angles of the three isosceles triangles and the rectangular triangle, we obtain

$$7\theta_3 + \frac{\pi}{2} + \pi = 4\pi,$$

which simplifies to

$$\theta_3 = \frac{5\pi}{14}$$

Correspondingly

$$x_3 = R \sin \frac{5\pi}{14} \approx 0.9R.$$

Generalizing the result, the limiting case for n reflections corresponds to the ray with incidence angle θ_N .

The trajectory of the reflected ray determines n isosceles triangles with common sides and a base angle of θ_N .

Also note that the ray's trajectory now forms half of a regular $2n+1$ -gon. Summing over the internal angles of the n isosceles triangles and the rectangular triangle determined by the incident ray, we obtain

$$(2n\theta_N + \pi) + \left(\frac{\pi}{2} + \theta_N\right) = (n+1)\pi,$$

$$(2n+1)\theta_N + \frac{3\pi}{2} = (n+1)\pi,$$

which simplifies to

$$\theta_N = \frac{2n-1}{4n+2}\pi$$

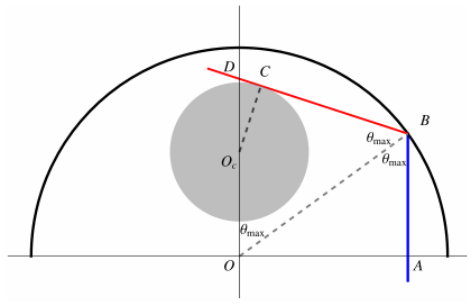
Correspondingly

$$x_n = R \sin \frac{2N-1}{4N+2}\pi = R \cos \frac{\pi}{2N+1}$$

T2-B [4.0 pts]. Solar Cooker

T2-B1 [2.0 pts].

The limiting case is shown in the figure below.



- O_c is the center of the container.
- A is the incidence point on the x -axis
- B is the reflection point
- The reflected ray is tangential to the container at the point C
- The reflected ray crosses the y -axis at D .

As the triangle ODB is isosceles,

$$\angle ODB = \pi - 2\theta_{\max},$$

$$\cos \theta_{\max} = \frac{R}{2\overline{OD}}.$$

$$\overline{OD} = \frac{R}{2 \cos \theta_{\max}}.$$

The triangle O_cDC is rectangle

$$\begin{aligned} a &= \overline{O_cD} \sin(\pi - 2\theta_{\max}) \\ &= \left(\overline{OD} - \frac{R}{2} \right) \sin(2\theta_{\max}) \\ &= \left(\frac{R}{2 \cos \theta_{\max}} - \frac{R}{2} \right) \sin(2\theta_{\max}) \\ &= R \sin \theta_{\max} - \frac{R}{2} \sin(2\theta_{\max}) \end{aligned}$$

The coefficients read

$$\boxed{\alpha = R}$$

$$\boxed{\beta = -\frac{R}{2}}$$

T2-B2 [1.5 pts]. Without the mirror:

$$P_0 = 2Ial,$$

where I is the sunlight intensity and l is the length of the cylinder.

With the mirror:

$$\begin{aligned} P &= 2I\overline{OA}l \\ &= 2IRl \sin \theta_{\max} \end{aligned}$$

Thus

$$\begin{aligned} \frac{P}{P_0} &= \frac{2IRl \sin \theta_{\max}}{2Ial} \\ &= \frac{R \sin \theta_{\max}}{R \sin \theta_{\max} (1 - \cos \theta_{\max})} \end{aligned}$$

$$\boxed{\frac{P}{P_0} = \frac{1}{1 - \cos \theta_{\max}}}.$$

Alternative answer in terms of α and β : Without the mirror:

$$P_0 = 2Ial = 2I(\alpha \sin \theta_{\max} + \beta \sin (2\theta_{\max}))l$$

where I is the sunlight intensity and l is the length of the cylinder.
With the mirror:

$$\begin{aligned} P &= 2I\overline{OAl} \\ &= 2IRl \sin \theta_{\max} \end{aligned}$$

Thus

$$\begin{aligned} \frac{P}{P_0} &= \frac{2IRl \sin \theta_{\max}}{2Ial} \\ &= \frac{R \sin \theta_{\max}}{\alpha \sin \theta_{\max} + \beta \sin (2\theta_{\max})} \end{aligned}$$

$$\frac{P}{P_0} = \frac{R}{\alpha + 2\beta \cos \theta_{\max}}.$$

T2-B3 [0.5 pts]. If $P = 5P_0$

$$\begin{aligned} 5 &= \frac{1}{1 - \cos \theta_{\max}}, \\ \cos \theta_{\max} &= \frac{4}{5}, \\ \sin \theta_{\max} &= \frac{3}{5}. \end{aligned}$$

Finally

$$a = (1\text{m})\frac{3}{5}\left(1 - \frac{4}{5}\right).$$

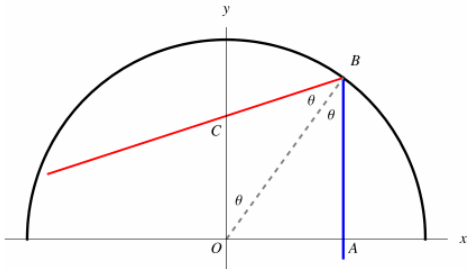
$a = 12\text{ cm.}$

Alternative solution parametrized in terms of α and β

$$a = -\frac{R\sqrt{100\beta^2 - (R - 5\alpha)^2}}{50\beta}$$

T2-C [4.5 pts]. Caustics and Cusp

T2-C1 [0.5 pts].



The reflected ray forms an angle 2θ with the vertical axis. Therefore, the slope of the line is

$m_A = \cot(2\theta).$

The reflected ray intersects the y -axis at C , $\overline{OC} = b_A$, forming the isosceles triangle OBC . from the intersection

$$\cos \theta = \frac{R}{2b_A}.$$

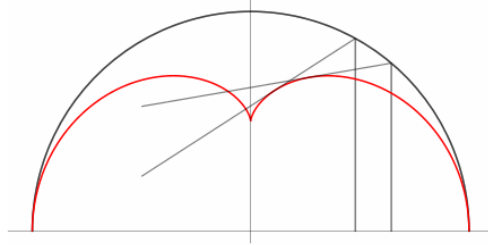
$$b_A = \frac{R}{2\cos \theta}.$$

Equivalent expressions:

$$m_A = \cot(2\theta) = \frac{1}{\tan(2\theta)} = \frac{\cot(\theta)}{2} - \frac{\tan(\theta)}{2}$$

$$b_A = \frac{R}{2 \cos \theta} = \frac{R}{2} \sec \theta$$

T2-C2 [2.0 pts].



The slope is

$$\begin{aligned} m_B &= \cot(2\theta + 2\Delta\theta) \\ &= \frac{\cot(2\theta) \cot(2\Delta\theta) - 1}{\cot(2\theta) + \cot(2\Delta\theta)} \\ &= \frac{\cot(2\theta) - \tan(2\Delta\theta)}{\cot(2\theta) \tan(2\Delta\theta) + 1} \\ &= (\cot(2\theta) - 2\Delta\theta)(1 - 2\Delta\theta \cot(2\theta)) + \dots \\ &= \cot(2\theta) - 2\Delta\theta(1 - \tan^2(2\Delta\theta)) + \dots \end{aligned}$$

Then, the first-order expansion of m_B reads

$$m_B = \cot(2\theta) - 2 \csc^2(2\theta) \Delta\theta.$$

On the other hand, the intercept is

$$\begin{aligned} b_B &= \frac{R}{2} \sec(\theta + \Delta\theta) \\ &= \frac{R}{2} \frac{\sec(\theta) \sec(\Delta\theta)}{1 - \tan(\theta) \tan(\Delta\theta)} \\ &= \frac{R}{2} (\sec(\theta)(1 + \tan(\theta)\Delta\theta)) + \dots \end{aligned}$$

and the first-order expansion reads

$$b_B = \frac{R}{2} \sec(\theta) (1 + \tan(\theta)\Delta\theta)$$

Equivalent expressions:

$$\begin{aligned} m_B &= \cot(2\theta) - 2 \csc^2(2\theta) \Delta\theta \\ &= \frac{1}{2} \csc^2(2\theta) (\sin(4\theta) - 4\Delta\theta) \\ &= \frac{1}{2} (\cot(\theta) - \tan(\theta) - \csc^2(\theta) \sec^2(\theta)) \Delta\theta \\ b_B &= \frac{R}{2} \sec(\theta) (1 + \tan(\theta)\Delta\theta) \\ &= \frac{R}{2 \cos(\theta)} (1 + \tan(\theta)\Delta\theta) \end{aligned}$$

T2-C3 [1.0 pts].

The coordinates of the intersection point, (X_C, Y_C) , fulfill the following equations:

$$\begin{aligned} Y_C &= m_A X_C + b_A \\ Y_C &= m_B X_C + b_B, \end{aligned}$$

Solving:

$$(X_C, Y_C) = \left(-\frac{b_A - b_B}{m_A - m_B}, \frac{m_A b_B - b_A m_B}{m_A - m_B} \right)$$

Replacing m_B and b_B , we find:

$$(X_C, Y_C) = \left(R \sin^3(\theta), \frac{R}{2} \cos(\theta)(2 - \cos(2\theta)) \right)$$

$$X_C = R \sin^3(\theta)$$

$$Y_C = \frac{R}{2} \cos(\theta)(2 - \cos(2\theta))$$

Equivalent expressions:

$$\begin{aligned} Y_C &= \frac{R}{2} \cos(\theta)(2 - \cos(2\theta)) \\ &= \frac{R}{4} (3 \cos(\theta) - \cos(3\theta)) \\ &= \frac{R}{4} (3 \cos(\theta) - \cos^3(\theta) + 3 \sin^2(\theta) \cos(\theta)) \end{aligned}$$

T2-C4 [1.0 pts]. For $\theta \ll 1$, we use the approximations

$$\begin{aligned} \sin \theta &\approx \theta, \\ \cos \theta &\approx 1 - \frac{\theta^2}{2}. \end{aligned}$$

Thus, near the cusp, the caustic is approximated as

$$\begin{aligned} (X_C, Y_C) &\approx \left(R\theta^3, \frac{R}{4}(2 + 3\theta^2) \right) \\ (X_C, Y_C) &\approx \left(R\theta^3, \frac{R}{2} + \frac{3R}{4}\theta^2 \right) \end{aligned}$$

Solving for θ , we get

$$\theta = \left(\frac{X_C}{R} \right)^{\frac{1}{3}},$$

and substitute in Y_C

$$Y_C = \frac{R}{2} + \frac{3R}{4} \left(\frac{X_C}{R} \right)^{\frac{2}{3}}$$

Comparing with

$$Y_C = v|X_C|^{p/q} + u,$$

we find

$$v = \frac{3}{4} R^{\frac{1}{3}}$$

$$u = \frac{R}{2},$$

$$p = 2$$

$$q = 3$$

This shows that the shape of the cusp is universal and that its tip is located at $y = \frac{R}{2}$.