# T1. Ferromagnetic domain walls dynamics

In this problem you will calculate physical quantities using the Ferromagnetic Simulator program. The detailed program description is provided at the end of the problem. Record the obtained numerical values in Excel files, specifying the number of the corresponding problem section in the title of each file. In each column indicate which physical quantity it contains. Graphs can be plotted in the same files. Include the necessary formulas for calculations and figures explaining your solution in the answer sheets. Save all the files generated during the solution to the "**report**" folder.

**Warning!** The program may run calculations for several minutes. To speed up your work, you can launch multiple instances of the program. You can analyze graphs obtained from one instance while another performs calculations. Remember to save your results so you can use them later.

Ferromagnets are materials that can have a nonzero magnetic moment even in the absence of an external magnetic field. The state of a ferromagnetic material is characterized by magnetic moment density $\vec{M}$, called magnetization. In equilibrium magnetic moment distribution corresponds to a local energy minimum. The ferromagnet is divided into regions with different magnetic moment orientations — domains. These domains are separated by domain walls — transition regions where the magnetization changes direction.

If the energy is not minimal, or if an external magnetic field is applied, the magnetization changes with time. The magnetization evolution is governed by the Landau-Lifshitz-Gilbert (LLG) equation:

$$\frac{\partial \vec{M}}{\partial t} = -\gamma \vec{M} \times \vec{B}_{\text{eff}} + \alpha \frac{\vec{M}}{M_0} \times \frac{\partial \vec{M}}{\partial t}.$$

The first term describes magnetization precession in magnetic field and the second term describes energy dissipation. Here $\vec{B}_{\text{eff}}$ is effective magnetic field acting at a given point. The effective field depends on magnetization at a given point and magnetization in its neighborhood. Here $\gamma$ is gyromagnetic ratio, (the ratio of the electron's magnetic moment to its angular momentum), $\alpha$ is dimensionless Gilbert damping constant, $M_0$ is magnetization magnitude.

Suppose that the ferromagnet energy density (i.e. energy per unit volume) is known as a function of magnetization $w(\vec{M})$. The effective magnetic field can be calculated as derivative

$$\vec{B}_{\text{eff}} = -\frac{\partial w}{\partial \vec{M}},$$

where derivative over a vector is a vector made of derivatives with respect to magnetization components

$$B_{\text{eff},x} = -\frac{\partial w}{\partial M_x}, \quad B_{\text{eff},y} = -\frac{\partial w}{\partial M_y}, \quad B_{\text{eff},z} = -\frac{\partial w}{\partial M_z}.$$

For instance, if material is placed into an external magnetic field $\vec{B}_0$, corresponding contribution to energy density is

$$\Delta w_B = -\vec{M} \cdot \vec{B}_0,$$

and corresponding effective magnetic field change is $\Delta \vec{B}_{\text{eff}} = \vec{B}_0$.

In this problem you may use the following formulas for vector product in Cartesian coordinates

$$\left(\vec{a} \times \vec{b}\right)_x = a_y b_z - a_z b_y,$$
$$\left(\vec{a} \times \vec{b}\right)_y = a_z b_x - a_x b_z,$$
$$\left(\vec{a} \times \vec{b}\right)_z = a_x b_y - a_y b_x.$$

Unit vectors along coordinate axes $x$, $y$ and $z$ are $\vec{e}_x$, $\vec{e}_y$, $\vec{e}_z$ respectively.

Parts A, B and C can be solved independently.

# Part A. Single magnetic moment

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| **A1** | Show that if magnetization $\vec{M}$ evolves according to LLG equation, vector $\vec{M}$ magnitude is constant over time. |

Suppose that magnetization is homogeneous (does not depend on coordinates). Define unit vector $\vec{m} = \vec{M}/M_0$, where $M_0 = |\vec{M}|$. The LLG equation can be written in the form

$$\frac{\mathrm{d}\vec{m}}{\mathrm{d}t} = -\gamma\vec{m} \times \vec{B}_{\text{eff}} + \alpha\vec{m} \times \frac{\mathrm{d}\vec{m}}{\mathrm{d}t}.$$

Magnetization depends only on time so we can write full derivative instead of partial derivative. For calculatio
it is useful to have an explicit formula for time derivative

$$\frac{\mathrm{d}\vec{m}}{\mathrm{d}t} = -\frac{\gamma}{1+\alpha^2}\vec{m} \times \vec{B}_{\text{eff}} - \frac{\alpha\gamma}{1+\alpha^2}\left((\vec{m} \cdot \vec{B}_{\text{eff}})\,\vec{m} - \vec{B}_{\text{eff}}\right).$$

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| **A2** | Suppose ferromagnetic is placed into homogeneous constant magnetic field $B_0$ directed along $z$ axis. Then $\vec{B}_{\text{eff}} = B_0\vec{e}_z$. Find the time dependence of the component $m_z(t)$. At the initial time moment $m_y = m_z = 0$, $m_x = 1$.<br>You may use the following integral<br>$$\int \frac{\mathrm{d}x}{1-x^2} = \frac{1}{2}\ln\left\|\frac{1+x}{1-x}\right\| + C.$$ |

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| **A3** | Consider the system described in the previous task. Show that magnetic moment $\vec{m}$ projection onto the $xy$ plane rotates around $z$ axis with constant angular velocity. Find this angular velocity $\Omega$, express answer in terms of $\gamma$, $\alpha$, $B_0$. |

Therefore magnetic moment precesses around the effective magnetic field $\vec{B}_{\text{eff}}$, and magnetic moment gradually gets aligned with magnetic field.

# Part B. Static domain wall

In this part we consider thin ferromagnetic wire with square section (square side size is $a$), wire length is $L$. Let us divide the wire into $N = L/a$ cubic cells with side length $a$. We will assume that within each cell, the magnetization $\vec{M}_i$ is uniform. Since the magnitude of the magnetization $M_0$ is constant, it is sufficient to specify the unit vector $\vec{m}_i = \vec{M}_i/M_0$. Let the $x$ axis be directed along the wire and $y$, $z$ axes are along the section sides.

Each cubic cell (with side $a$) energy consists of the following terms:

1. Anisotropy energy, which depends on magnetization direction. For the considered wire

$$W_{an} = -\frac{Ka^3M_z^2}{2M_0^2} + \frac{K_1a^3M_y^2}{2M_0^2},$$

where $K, K_1 > 0$ are constants. This energy is minimal when magnetization is directed along $z$ axis.

2. Exchange energy, which depends on relative orientation of neighbouring pair of cell magnetic moments. For one pair of cells the energy is

$$W_{ex,i} = -\frac{A}{M_0^2}\vec{M}_i\vec{M}_{i+1}a,$$

where $A > 0$ is constant, therefore configuration with magnetic moments aligned in the same direction is preferred. There is such contribution for each pair of neighboring cells.

3. Energy in external magnetic field

$$W_B = -a^3\vec{B}\vec{M}.$$

Note that the expressions above are for energy, not energy density. Thus for **calculating effective magnetic field** $\vec{B}_{\text{eff}}$ projections one should differentiate the energy with respect to magnetization and **divide by** $a^3$.

Consider the material with following parameters:

1. $A = 4.5 \times 10^{-10}\ \frac{\mathrm{J}}{\mathrm{m}}$;
2. $K = 5.0 \cdot 10^5\ \frac{\mathrm{J}}{\mathrm{m}^3}$;
3. $\gamma = 1.76 \cdot 10^{11}\ \frac{1}{\mathrm{s \cdot T}}$;
4. $M_0 = 6.0 \cdot 10^5\ \frac{\mathrm{A}}{\mathrm{m}}$.

Cubic cell side size is $a = 3.0\ \mathrm{nm}$.

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| **B1** | Effective magnetic field corresponding to the energy above can be written as $$\vec{B}_{\text{eff},i} = B_{\text{an}}m_{i,z}\vec{e}_z + B_{\text{ex}}(\vec{m}_{i-1} + \vec{m}_{i+1}) - k_1 B_{\text{an}}m_{i,y}\vec{e}_y + \vec{B}_0.$$ Here $k_1 = K_1/K$ is a dimensionless coefficient. In all calculations we use $k_1 = 1$. Find formulas and numerical values for fields $B_{\text{ex}}$ and $B_{\text{an}}$. Calculate characteristic frequency of magnetic moment precession $\omega = \gamma B_{\text{an}}$. |

Below all magnetic fields are measured in units of $B_{\text{an}}$, time is measured in units of $1/\omega$, $\tau = \omega t$. Thus dimensionless effective field is ($\beta = B_{\text{ex}}/B_{\text{an}}$, $\vec{b} = \vec{B}_0/B_{\text{an}}$)

$$\vec{b}_{\text{eff},i} = m_{i,z}\vec{e}_z + \beta(\vec{m}_{i-1} + \vec{m}_{i+1}) - k_1 m_{i,y}\vec{e}_y + \vec{b},$$

and LLG equation is

$$\frac{\mathrm{d}\vec{m}_i}{\mathrm{d}\tau} = -\vec{m}_i \times \vec{b}_{\text{eff}} + \alpha\vec{m}_i \times \frac{\mathrm{d}\vec{m}_i}{\mathrm{d}\tau}.$$

Anisotropy energy is minimal when magnetic moments are collinear with $z$ axis, $m_{i,z} = \pm 1$. Domain wall between regions with $m_{i,z} = +1$ and $m_{i,z} = -1$ is a transitional region, where magnetization direction changes from positive $z$ axis direction to negative $z$ axis direction. You will use **Ferromagnetic Simulator** program to study magnetization dependence on coordinate $x$ inside the domain wall.

At the initial moment of time in the left part of the wire magnetization is $m_{i,z} = +1$, while in the right part $m_{i,z} = -1$. Abrupt change in magnetization direction leads to large exchange energy, therefore this configuration is not optimal. Let us set a large dissipation constant $\alpha$ and consider how system changes over time. After relatively short time period the system energy becomes close to minimal and the system almost stops changing.

**In parts B2-B4 external magnetic field is zero, $b = 0$!**

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| **B2** | In the program set parameters $\beta = 100$, $b = 0$. Choose time of evolution $T$ and damping constant in such a way that after time $T$ system change speed is negligible. Indicate used values of $\alpha$, $T$ in the answer sheets. In"report" folder save the program output for the chosen values of parameters $\alpha$ and $T$, choose file name "B2". Domain wall width $d$ is distance between points with $m_z = \pm 0.5$. Find the value of $d$ in the $a$ units. |

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| **B3** | Study the dependence of domain wall width on the exchange energy (proportional to value of $\beta$) in the interval from $\beta = 20$ to $\beta = 400$. Obtain at least 5 points. Write the values of $\beta$ you used and corresponding values of $d$ and $\varphi$ into an Excel file "B3". Remember to write the physical quantities names for the columns you used. Save to "report" folder the program output with file name "B3_j", where $j$ is a measurement number. Provide necessary calculations and explanation in working sheets. |

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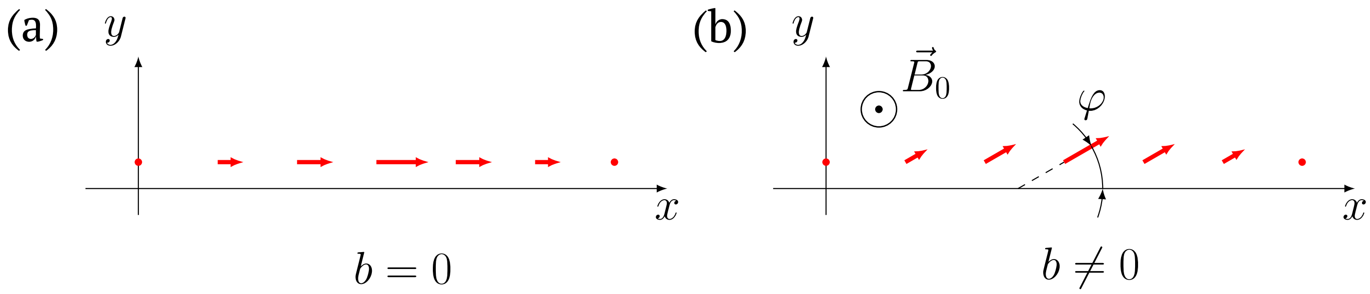
| **B4** | If $d(\beta) \sim \beta^c$, calculate value of $c$ using results from the previous part. If you need to plot a graph, save it to Excel file "B3". |

# Part C. Domain wall motion in external field




*Magnetic moment projection on $xy$ plane dependence on $x$ coordinate:*
*(a) without external magnetic field, (b) in external field.*

Switch on external magnetic field along $z$ axis, $b_{0,z} = b > 0$. Domain wall starts moving. If $b < b_{\text{cr}}$, domain wall settles into uniform motion, and for $b > b_{\text{cr}}$ the motion stays non-uniform. Critical magnetic field value $b_{\text{cr}}$ is called Walker limit.

Magnetic moment distribution in moving domain wall can be related to that of static domain wall. Without magnetic field $m_x(x) = f(x - x_0)$, $m_y = 0$, where $x_0$ is domain wall center coordinate. Then after magnetic field is turned on $m_x(x) = f(x - x_0)\cos\varphi$, $m_y = f(x - x_0)\sin\varphi$, where angle $\varphi$ does not depend on coordinate and $x_0$ is domain wall center coordinate at a given time. During uniform motion angle $\varphi$ does not depend on time.

**In part C all the lengths are measured in units of $a$, and all the times in units of $1/\omega$, always use values $\beta = 100$, $\alpha = 0.2$.**

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| **C1** | In which direction the wall starts moving (to the right or to the left)? |

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| **C2** | In program set the values $\alpha = 0.2$, $\beta = 100$. Obtain the dependence of domain wall velocity $v$ and angle $\varphi$ on external field $b$. Study the magnetic fields in range from $b = 0$ to $b = b_{\text{cr}}$. Write used values $b$ and corresponding values of $d$ and $\varphi$ in Excel file "C2". Remember to write the physical quantities names for the columns you used. Save to "report" folder the program output with file name "C2_j", where $j$ is a measurement number. Provide necessary calculations and explanation in working sheets. It is known that $b_{\text{cr}} < 0.15$. You should obtain at least 5 points in the range $b < b_{\text{cr}}$. Note that if during the domain wall motion distance from its center to the wire edge becomes less than 40, the edge effects modify the solution significantly and you will not be able to obtain correct results from it. Thus for field values close to critical you may need to use the maximum available length $L = 200$. For smaller field values you may use shorter length to speed up calculations. |

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| **C3** | Plot the dependence $v(b)$, calculate the slope $\frac{\mathrm{d}v}{\mathrm{d}b}$ for the linear part. For plotting use Excel file "C2". |

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| **C4** | It follows from theoretical calculations that for $b < b_{\text{cr}}$ the velocity is equal to<br>$$v = v_c \sin 2\varphi.$$<br>Using data from part C2, plot a linearized graph and find values of $v_c$ and $b_{\text{cr}}$. For plotting use Excel file "C2". |

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| **C5** | Using numerical values from part B, calculate $v_c$ in $\mathrm{m/s}$, and $B_{\text{cr}} = B_{\text{an}} b_{\text{cr}}$ in T. |

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| **C6** | Let us study domain wall motion for $b > b_{\text{cr}}$. For $b = 0.2$ measure the dependence of domain wall coordinate on time. Write domain wall coordinates and corresponding time values in Excel file "C6", remember to write names of physical quantities in corresponding columns. Plot this dependence. Save the program output you used to "report" folder in file "C6". |

# Instructions for Ferromagnetic Simulator program

The Ferromagnetic Simulator program numerically solves the Landau-Lifshitz-Gilbert (LLG) equation for a system of magnetic moments described in Part B. By toggling the radio buttons at the top, you may select the problem section (B or C). It affects the number of cubes (N) and the initial magnetic moment values.

**Specifying parameters** For both problem sections the following parameters can be specified :

1. Gilbert damping coefficient ($\alpha$) (input field: *alpha*), the input value must satisfy $\alpha \in [0; 1]$.
2. Parameter $\beta$ (input field: *beta*), the input value must satisfy $\beta \in [0; 500]$.
3. External magnetic field ($b$) (input field: *b*), the input value must satisfy $b \in [0, 1]$.
4. Simulation time ($T$) (input field:*T*, in dimensionless units, as defined in Part B), the input value must satisfy $T \in [0, 100]$. The program computes the solution over the interval $t \in [0, T]$.

In Part C, you may also specify the distance ($L$) from the initial position of the domain wall center to the right edge of the wire, measured in units of the cubic cell size ($a$). The input value must satisfy $L \in [50, 200]$. Increasing $L$ improves the observation of domain wall motion but increases computation time. The domain wall is always positioned at a fixed distance $L_1 = 50$ from the left edge, so the total system length is $L_1 + L$. **Running the Simulation** Set the desired parameters and click **Solve**. The program will numerically integrate the LLG equation. For some initial conditions, computation may take several minutes. Once the solution is computed, the **Plot** button becomes active. **Plotting the Results** Specify a time $t \in [0, T]$ and click **Plot**. The program will display smoothed curves of $m_x(x)$, $m_y(x)$ and $m_z(x)$ at time $t$. Note that the actual computed values are only at integer positions $x = i$. The program yields solutions at **200 time points**, equally spaced at intervals of $T/200$. **If a requested time does not match, it is rounded to the nearest available value. Save/Load Functionality** Use **Save** to store solutions in the "report" folder. Use **Load** to retrieve previously saved solutions. Once loaded, you can plot $m_x(x)$, $m_y(x)$, and $m_z(x)$ for any of the 200 saved time points without recomputing. Note: save all the solutions you use. **Initial conditions** In Part B the program assumes $N = 201$ and the following initial conditions

1. $m_{i,z} = +1$ for $i = 0, \ldots, 97$,
2. $m_{i,x} = 1$ for $i = 98, \ldots, 102$,
3. $m_{i,z} = -1$ for $i = 103, \ldots, 200$.

All other components of $\vec{m}_i$ are zero.

As an initial state for part C the program precomputes static solution for $\beta = 100$ with $N = 401$ cubic cells. To match the specified system length (determined by parameter $L$), excess values beyond the required domain are truncated.

# Solution

**A1.** The length of $\vec{M}$ squared is equal to

$$M^2 = \vec{M} \cdot \vec{M},$$

thus, its time derivative is

$$\frac{\partial}{\partial t} M^2 = 2\vec{M} \cdot \frac{\partial \vec{M}}{\partial t}.$$

For any vectors $\vec{a}, \vec{b}$, their vector product $\vec{a} \times \vec{b}$ is orthogonal to the vectors. In particular, vectors $\vec{M} \times \vec{B}_{\text{eff}}$ and $\vec{M} \times \frac{\partial \vec{M}}{\partial t}$ are orthogonal to $\vec{M}$, so according to LLG equation $\vec{M}$ and $\frac{\partial \vec{M}}{\partial t}$ are perpendicular, therefore

$$\vec{M} \cdot \frac{\partial \vec{M}}{\partial t} = 0, \quad \frac{\partial}{\partial t} M^2 = 0,$$

and the magnitude of $\vec{M}$ is constant.

$$\boxed{\frac{\partial}{\partial t} M^2 = 2\vec{M} \cdot \frac{\partial \vec{M}}{\partial t} = 0}$$

**A2.** Consider the $z$-component of the equation

$$\frac{\mathrm{d}\vec{m}}{\mathrm{d}t} = -\frac{\gamma}{1+\alpha^2} \vec{m} \times \vec{B}_{\text{eff}} - \frac{\alpha\gamma}{1+\alpha^2} \left( (\vec{m} \cdot \vec{B}_{\text{eff}}) \, \vec{m} - \vec{B}_{\text{eff}} \right).$$

The vector product $(\vec{m} \times \vec{B}_{\text{eff}})$ is perpendicular to $z$ axis, so only the second term contributes to the equation. Taking into account $\vec{m} \cdot \vec{B}_{\text{eff}} = m_z B_0$ we get

$$\dot{m}_z = \frac{\alpha\gamma}{1+\alpha^2} B_0 (1 - m_z^2).$$

Separating variables

$$\frac{\mathrm{d}m_z}{1 - m_z^2} = \frac{\alpha\gamma}{1+\alpha^2} B_0 \, \mathrm{d}t$$

and integrating the resulting expression from $0$ to $t$ we obtain

$$\frac{1}{2} \ln \left( \frac{1 + m_z(t)}{1 - m_z(t)} \right) \Bigg|_0^t = \frac{\alpha\gamma}{1+\alpha^2} B_0 t.$$

Substituting initial condition $m_z(0) = 0$,

$$\frac{1}{2} \ln \left( \frac{1 + m_z(t)}{1 - m_z(t)} \right) = \frac{\alpha\gamma}{1+\alpha^2} B_0 t,$$

we find the answer

$$\boxed{m_z(t) = \frac{\exp\left(\frac{\alpha\gamma B_0 t}{1+\alpha^2}\right) - \exp\left(-\frac{\alpha\gamma B_0 t}{1+\alpha^2}\right)}{\exp\left(\frac{\alpha\gamma B_0 t}{1+\alpha^2}\right) + \exp\left(-\frac{\alpha\gamma B_0 t}{1+\alpha^2}\right)} = \tanh\left(\frac{\alpha\gamma B_0 t}{1+\alpha^2}\right)}$$

## A3. The first solution

Denote by $\vec{m}_{xy}$ the projection of the vector $\vec{m}$ onto the $xy$-plane. The vector $\vec{B}_{\text{eff}}$ is directed along the $z$ axis, so that its projection onto the $xy$-plane is zero. Next, $\vec{m} \cdot \vec{B}_{\text{eff}} = m_z B_0$ and

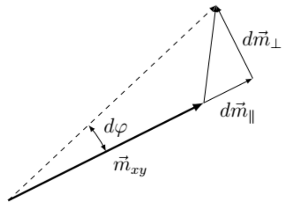
$$\vec{m} \times \vec{B}_{\text{eff}} = (\vec{m}_{xy} + m_z \vec{e}_z) \times \vec{B}_{\text{eff}} = \vec{m}_{xy} \times \vec{B}_{\text{eff}}, \quad (\vec{e}_z \times \vec{B}_{\text{eff}} = 0),$$

so that the projection of the LLG equation onto the $xy$-plane takes the form

$$\frac{\mathrm{d}\vec{m}_{xy}}{\mathrm{d}t} = -\frac{\gamma}{1+\alpha^2}\vec{m}_{xy} \times \vec{B}_{\text{eff}} - \frac{\alpha\gamma}{1+\alpha^2} m_z B_0 \, \vec{m}_{xy}.$$

Here the first term (with magnitude $\gamma B_0 m_{xy}/(1+\alpha^2)$) is orthogonal to $\vec{m}_{xy}$ and hence leads to the rotation of $\vec{m}_{xy}$. The second term is parallel to $\vec{m}_{xy}$ and leads to the length change.

During the time period $\mathrm{d}t$, a perpendicular change component is




$$\mathrm{d}m_\perp = \frac{\gamma B_0 m_{xy}}{1+\alpha^2}\mathrm{d}t,$$

so that the rotation angle is $\mathrm{d}\varphi = \mathrm{d}m_\perp / m_{xy}$, and the angular velocity is

$$\Omega = \frac{\mathrm{d}\varphi}{\mathrm{d}t} = \frac{\gamma B_0}{1+\alpha^2}.$$

Another way to obtain $\Omega$ is to notice that if some vector rotates with an angular velocity $\vec{\Omega}$, then its time derivative is equal to

$$\frac{\mathrm{d}\vec{m}_{xy}}{\mathrm{d}t} = \vec{\Omega} \times \vec{m}_{xy}.$$

Comparing with the first term in the LLG equation, we immediately get

$$\vec{\Omega} = \frac{\gamma \vec{B}_{\text{eff}}}{1+\alpha^2} = \frac{\gamma B_0}{1+\alpha^2}\vec{e}_z.$$

So the answer is

$$\boxed{\Omega = \frac{\gamma B_0}{1+\alpha^2}}$$

## A3. The second solution

Project the LLG equation onto the $x$ and $y$ axis:

$$\begin{cases} \dot{m}_x = -\dfrac{\gamma B_0}{1+\alpha^2} m_y - \dfrac{\alpha\gamma B_0 m_z}{1+\alpha^2} m_x, \\[2ex] \dot{m}_y = \dfrac{\gamma B_0}{1+\alpha^2} m_x - \dfrac{\alpha\gamma B_0 m_z}{1+\alpha^2} m_y. \end{cases}$$

Introduce a new complex variable $\hat{m} = m_x + i m_y$ and take a sum of two equations above, where the second equation is multiplied by $i$ firstly:

$$\frac{\mathrm{d}}{\mathrm{d}t}(m_x + i m_y) = -\frac{\gamma B_0}{1+\alpha^2}(-m_y + i m_x) - \frac{\alpha\gamma m_z B_0}{1+\alpha^2}(m_x + i m_y).$$

The resulting equation can be rewritten as

$$\frac{\mathrm{d}\hat{m}}{\mathrm{d}t} = i\frac{\gamma B_0}{1+\alpha^2}\hat{m} - \frac{\alpha\gamma B_0 m_z}{1+\alpha^2}\hat{m}.$$

Separate variables and get

$$\int \frac{\mathrm{d}\hat{m}}{\hat{m}} = \int \mathrm{d}t \left( \frac{i\gamma B_0}{1+\alpha^2} - \frac{\alpha\gamma B_0}{1+\alpha^2} m_z \right).$$

Due to the initial condition $\hat{m}(0) = m_x(0) + i m_y(0) = 1$ we have

$$\ln \hat{m} = \frac{i\gamma B_0 t}{1+\alpha^2} - \frac{\alpha\gamma B_0}{1+\alpha^2} \int_0^t \mathrm{d}t \, m_z(t), \quad \hat{m} = A(t) e^{i\varphi(t)},$$

where the expression

$$A(t) = \exp\left( -\frac{\alpha\gamma B_0}{1+\alpha^2} \int_0^t m_z(t) \, \mathrm{d}t \right)$$

is the magnitude of $\vec{m}_{xy}$, and the expression

$$\varphi = \frac{\gamma B_0 t}{1+\alpha^2}$$

corresponds to an angle of rotation of $\vec{m}_{xy}$ since

$$m_x(t) = A(t) \cos\varphi(t), \quad m_y(t) = A(t) \sin\varphi(t).$$

The angle $\varphi$ is a linear function in $t$, so that the angular velocity is constant, and its value coincides with the one obtained earlier. Using the explicit formula for $m_z(t)$ one can find the magnitude $A(t)$. Indeed,

$$\ln A(t) = -\frac{\alpha\gamma B_0}{1+\alpha^2} \int_0^t m_z(t) \, \mathrm{d}t = -\int_0^t \frac{\alpha\gamma B_0 \mathrm{d}t}{1+\alpha^2} \tanh \frac{\alpha\gamma B_0 t}{1+\alpha^2} = -\ln\cosh \frac{\alpha\gamma B_0 t}{1+\alpha^2},$$

what implies

$$A(t) = \frac{1}{\cosh \dfrac{\alpha\gamma B_0 t}{1+\alpha^2}},$$

so that $m_z^2 + m_{xy}^2 = 1$, i.e. the magnitude of $\vec{m}$ is constant and equals $1$.

Here we use the formula

$$\int \tanh x \, \mathrm{d}x = \int \frac{\sinh x}{\cosh x} \mathrm{d}x = \int \frac{\mathrm{d}(\cosh x)}{\cosh x} = \ln\cosh x + C,$$

and the relation

$$\tanh^2 x + \frac{1}{\cosh^2 x} = 1.$$

One can avoid the use of complex numbers and find the solution in the form

$$m_x = A(t) \cos\varphi(t), \quad m_y(t) = A(t) \sin\varphi(t).$$

Substitute these expressions to the equations for $\dot{m}_x, \dot{m}_y$ and get

$$\frac{\mathrm{d}A}{\mathrm{d}t} \cos\varphi - A \sin\varphi \frac{\mathrm{d}\varphi}{\mathrm{d}t} = -\frac{\gamma B_0}{1+\alpha^2} A \sin\varphi - \frac{\alpha\gamma B_0 m_z}{1+\alpha^2} A \cos\varphi;$$

$$\frac{\mathrm{d}A}{\mathrm{d}t} \sin\varphi + A \cos\varphi \frac{\mathrm{d}\varphi}{\mathrm{d}t} = \frac{\gamma B_0}{1+\alpha^2} A \cos\varphi - \frac{\alpha\gamma B_0 m_z}{1+\alpha^2} A \sin\varphi.$$

From these equations one can get the time derivatives of the magnitude and the angle:

$$\frac{\mathrm{d}\varphi}{\mathrm{d}t} = \frac{\gamma B_0}{1+\alpha^2}, \quad \frac{\mathrm{d}A}{\mathrm{d}t} = -\frac{\alpha\gamma B_0 m_z}{1+\alpha^2}A.$$

Note that these relations are nothing but the equalities of the coefficients at $\cos\varphi$ and $\sin\varphi$ in the initial system of equations. The first equation gives a former expression for an angular velocity, and the second one — for the magnitude derivative.

$$\boxed{\Omega = \frac{\gamma B_0}{1+\alpha^2}}$$

**B1**. By definition

$$\vec{B}_{\text{eff},i} = -\frac{1}{a^3}\frac{\partial W(\vec{M}_i)}{\partial \vec{M}_i},$$

where $W$ is the energy of the system. It consists of contributions given in the task:

$$W = \sum_i a^3\left(-\frac{KM_{i,z}^2}{2M_0^2} + \frac{K_1 M_{i,y}^2}{2M_0^2}\right) - \sum_i A\vec{M}_i\vec{M}_{i+1}\frac{a}{M_0^2} - \sum_i a^3\vec{B}_0\vec{M}_i.$$

Here the first term represents anisotropy energy, only one term in it depends on a magnetization of every cell, so the corresponding magnetic field has the form

$$\vec{B}_{an,i} = \frac{\partial}{\partial M_{z,i}}\left(\frac{KM_{i,z}^2}{2M_0^2}\right)\vec{e}_z - \frac{\partial}{\partial M_{y,i}}\left(\frac{K_1 M_{i,y}^2}{2M_0^2}\right)\vec{e}_y = \frac{KM_{i,z}}{M_0^2}\vec{e}_z - \frac{k_1 K M_{i,y}}{M_0^2}\vec{e}_y = \frac{K}{M_0}\left(m_{i,z}\vec{e}_z - k_1 m_{i,y}\vec{e}_y\right).$$

Comparing with the formula from the task we can find

$$B_{\text{an}} = \frac{K}{M_0}.$$

The second term in $W$ represents the exchange energy, two terms in it depend on $\vec{M}_i$

$$-\frac{Aa}{M_0^2}\vec{M}_i(\vec{M}_{i-1} + \vec{M}_{i+1}),$$

so the effective magnetic field is

$$\vec{B}_{\text{ex}} = -\frac{1}{a^3}\frac{\partial}{\partial\vec{M}_i}\left(-\frac{Aa}{M_0^2}\vec{M}_i(\vec{M}_{i-1}+\vec{M}_{i+1})\right) = \frac{A}{a^2M_0^2}(\vec{M}_{i-1}+\vec{M}_{i+1}) = \frac{A}{a^2M_0}(\vec{m}_{i-1}+\vec{m}_{i+1}).$$

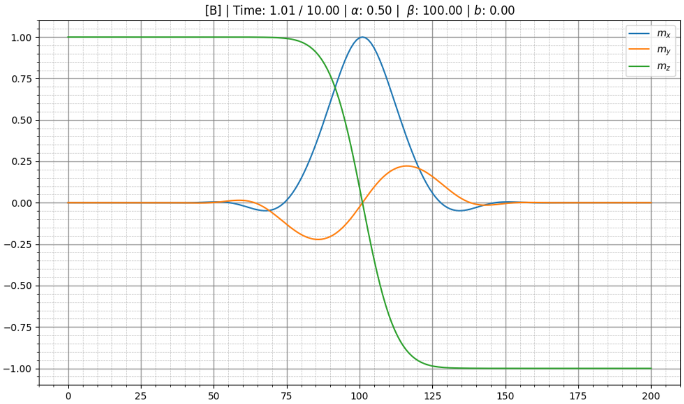
Therefore

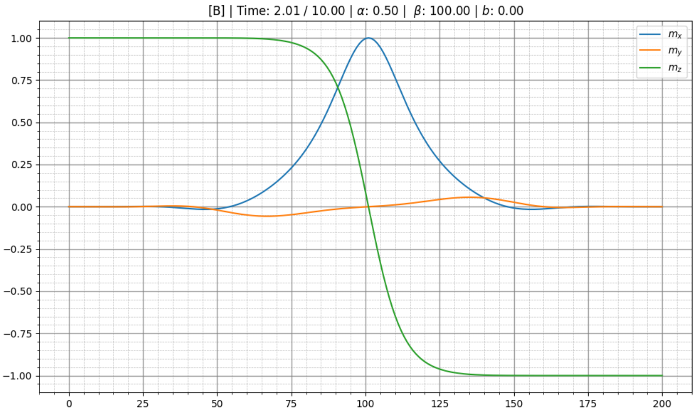
$$B_{\text{ex}} = \frac{A}{a^2M_0}.$$

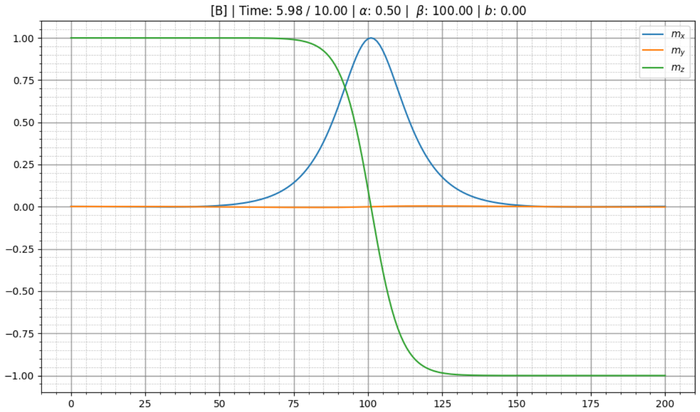
Differentiating the last term, we get a contribution to the effective magnetic field from the external magnetic field $\vec{B}_0$ as it is stated in the task.

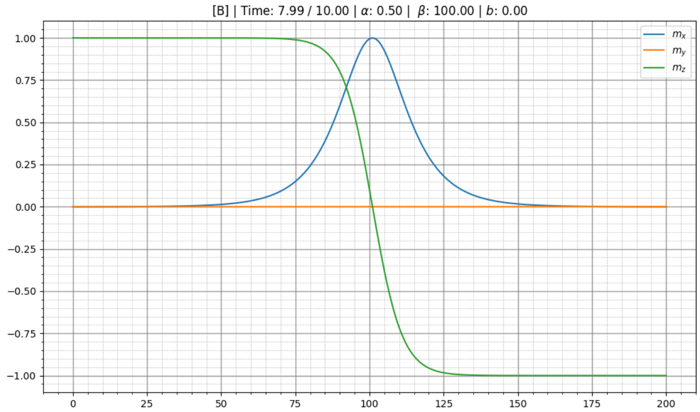
$$\boxed{B_{\text{an}} = \frac{K}{M_0} \approx 0.83\ \text{T}; \qquad B_{\text{ex}} = \frac{A}{M_0 a^2} \approx 83.33\ \text{T}; \qquad \omega \approx 1.47\cdot 10^{11}\ \text{s}^{-1}.}$$

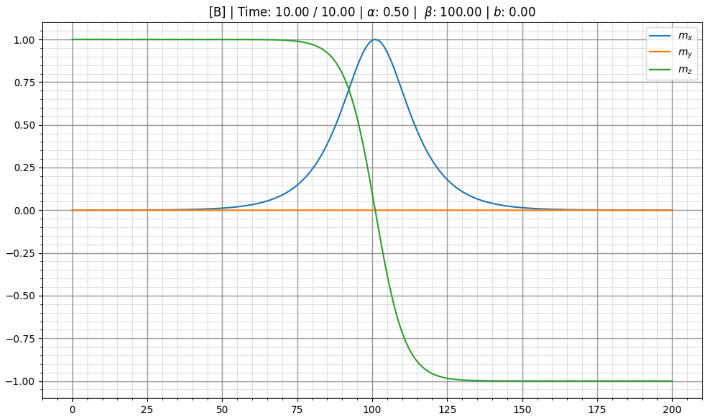
**B2**. Let us run the program for $\alpha = 0.5$, $\beta = 100$, $b = 0$, $T = 10$. The pictures show graphs of $m_x, m_y, m_z$ for different $t$ in range $[0, 10]$. Note that the magnetization distribution on the last two pictures is approximately the same, so we can consider it stationary. In particular, one of the features of the stationary state regime is that $m_y(x) = 0$ for all $x$.



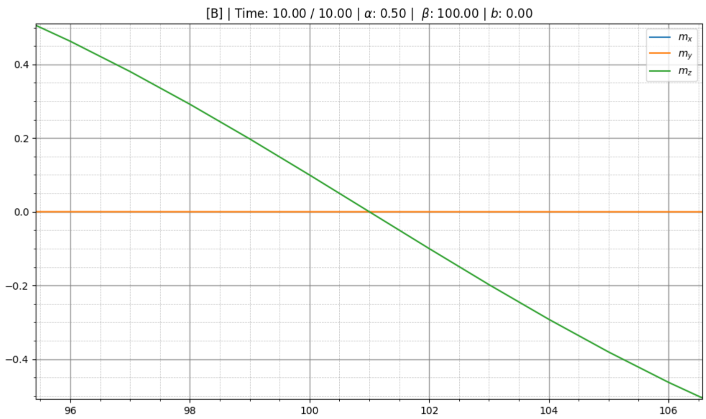




To increase an accuracy of $d$ measurement let us zoom in the part of the plot $m_z(x)$ for $t = 10$ in such a way that the side points correspond to $m_z = \pm 0.5$. The $x$-coordinates of the points with $m_z = 0.5$ and $m_z = -0.5$ are equal to $x_{\text{left}} = 95.5$ and $x_{\text{right}} = 106.5$ respectively, so that $d = x_{\text{right}} - x_{\text{left}} = 11.0$.



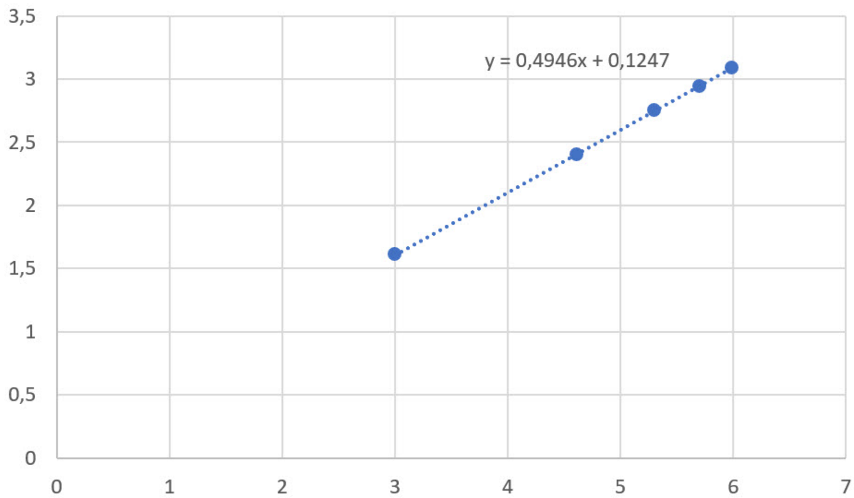

$$\boxed{d = 11.0}$$

**B3**. In part B2 we have seen that for $\alpha = 0.5$, $\beta = 100$, $b = 0$ and $T = 10$, the steady state regime occurs in the end. Thus, in the present part we will use the same values of the parameters $\alpha$ and $T$, and different $\beta \in [20, 400]$. We will find the width $d$ in the way described in the solution of B2, for $t = 10$. To ensure that the system is indeed in stationary state, we check that $m_y \equiv 0$ at $t = 10$ on the graph. Denote by $x_{\text{left}}$, (resp. $x_{\text{right}}$) the coordinates of the left (resp. right) borders of the domain wall. The width of the wall can be computed using the formula $d = x_{\text{right}} - x_{\text{left}}$.

| $\beta$ | $x_{\text{left}}$ | $x_{\text{right}}$ | $d$ |
|---|---|---|---|
| 20 | 98.5 | 103.5 | 5.0 |
| 100 | 95.5 | 106.5 | 11.0 |
| 200 | 93.2 | 108.8 | 15.6 |
| 300 | 91.5 | 110.5 | 19.0 |
| 400 | 90.0 | 112.0 | 22.0 |

**B4**. Let $d(\beta) = \gamma\beta^c$, where $\gamma$ is a constant coefficient. Then $\ln d(\beta) = \ln\gamma + c\ln\beta$, so that the dependence $\ln d\,(\ln\beta)$ is linear, and the slope is equal to $c$. Calculate $\log d$ and $\log\beta$ for the points obtained in B3, and then draw a graph in Excel using the resulting data.

| $\ln\beta$ | $\ln d$ |
|---|---|
| 3.00 | 1.61 |
| 4.61 | 2.40 |
| 5.30 | 2.75 |
| 5.70 | 2.94 |
| 5.99 | 3.09 |




*The plot* $\ln d\,(\ln\beta)$.

$$\boxed{c = 0.495}$$

**C1**. Set some $b \neq 0$ and run the program. We see that the domain wall moves to the right.

**To the right.**

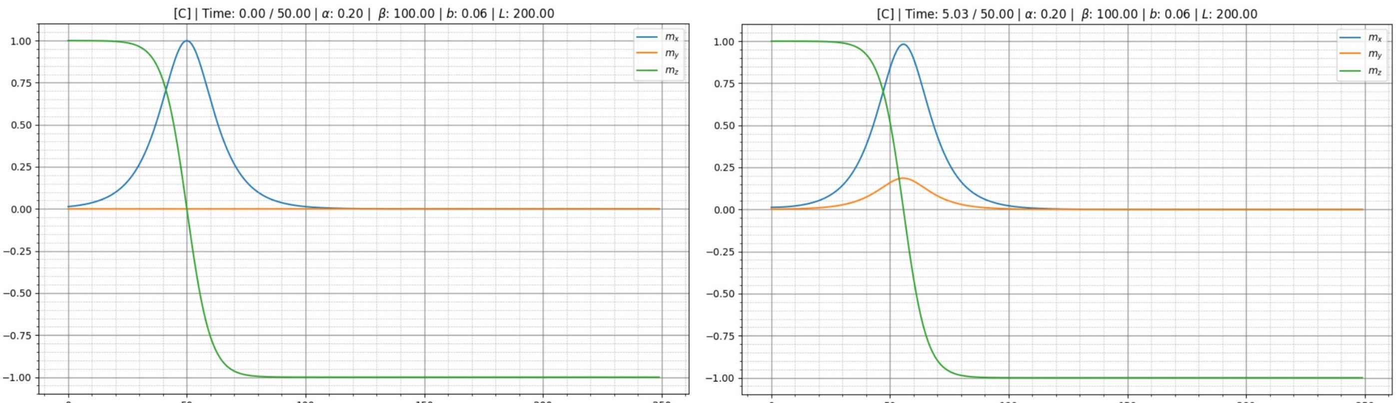
**C2**. After the start of the program, the transient process begins. For small enough values of $b$ the system settles into uniform motion, and the angle of magnetic fields rotation $\varphi$ stops changing. The evolution time should be large enough so that the motion becomes uniform. The length $L$ should be such that the domain wall does not reach the right side during the evolution time. On the other hand, with the growth of $T$ and $L$, the runtime increases, so that one should find optimal parameters.

Since just after the start of the program the uniform motion is not established yet, the velocity can not be computed using the formula $v = \frac{x-x_0}{t}$, where $x_0$ is the initial position of the center of the domain wall. The measurements should be taken in a steady state regime, so that one needs to determine the duration of the transient process. To do this one need to measure the dependence $x(t)$ for some $b$, and estimate the typical transient time. Another way to get correct measurements is to take them only when the shape of $m_y$ stops changing over time (one can see the change of $m_y$ only after zooming the plot). In a steady state regime, due to a high accuracy of measurements, it is enough to compute the velocity of the center of the wall by the formula $v = \frac{x_2-x_1}{t_2-t_1}$, i.e. using only two points.

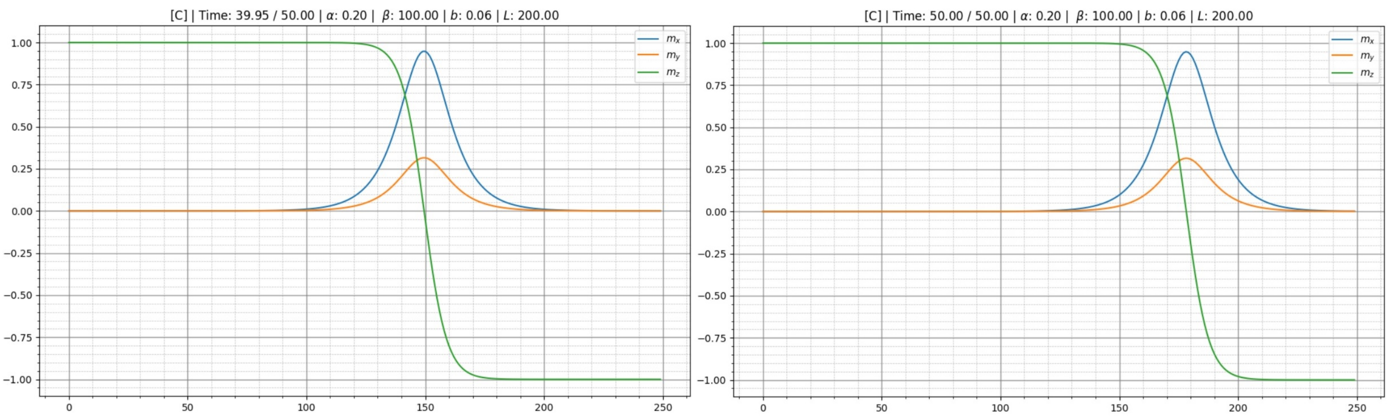
In the center of the domain wall we have $m_z = 0$, and both $m_x = m_{x0}$, $m_y = m_{y0}$ are maximal. (One can see from the plots that the coordinates of the maxima of $m_x$ and $m_y$ and the coordinate $x$ such that $m_z(x) = 0$, almost coincide). Therefore, we can find the angle $\varphi$ using either of the equivalent relations $m_{x0} = \cos\varphi,\ m_{y0} = \sin\varphi$, but only when the uniform motion is already established. Thus,

$$\varphi = \arcsin m_{y0} = \arccos m_{x0} = \arctan \frac{m_{y0}}{m_{x0}}.$$

If one constructs the graph of $m_y/m_x$, it can be seen that $\varphi$ indeed does not depend on the coordinate (participants are not required to do such a verification).




*Transient process: $m_y$ increases.*



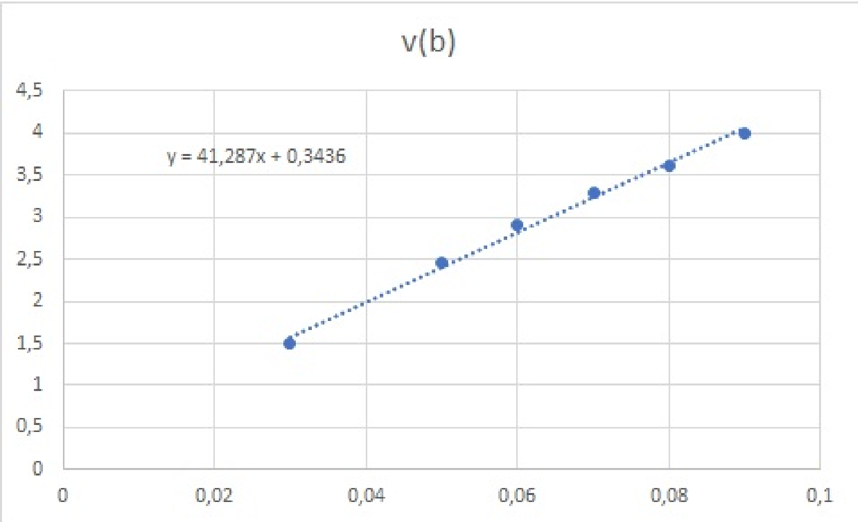

*Steady state regime: $m_y$ shape is constant.*

If $b > 0.1$, the domain wall behavior changes. Magnetization angle $\varphi$ increases, magnetic moment rotates around $z$ axis. The domain wall motion is accompanied by its center oscillations. Unlike the case $b < 0.1$, $m_y$ is sometimes greater than $m_x$ (orange and blue graphs swap). Hence $b_{\text{cr}} \approx 0.1$.

Let us measure $v(b)$ and $\varphi(b)$ for $b < 0.1$.

| $b$ | $m_y$ | $t_1$ | $x_1$ | $t_2$ | $x_2$ | $v$ | $\varphi$ |
|---|---|---|---|---|---|---|---|
| 0.03 | 0.152 | 40 | 101,5 | 50 | 116,5 | 1,50 | 0,152 |
| 0.05 | 0.258 | 40 | 134,0 | 50 | 158,5 | 2,45 | 0,261 |
| 0.06 | 0.316 | 40 | 149,0 | 50 | 178,0 | 2,90 | 0,322 |
| 0.07 | 0.378 | 40 | 164,0 | 50 | 197,0 | 3,30 | 0,387 |
| 0.08 | 0.446 | 35 | 159,0 | 45 | 195,0 | 3,60 | 0,462 |
| 0.09 | 0.525 | 37 | 176,0 | 43 | 200,0 | 4,00 | 0,893 |

**C3**. Plot the graph $v(b)$.

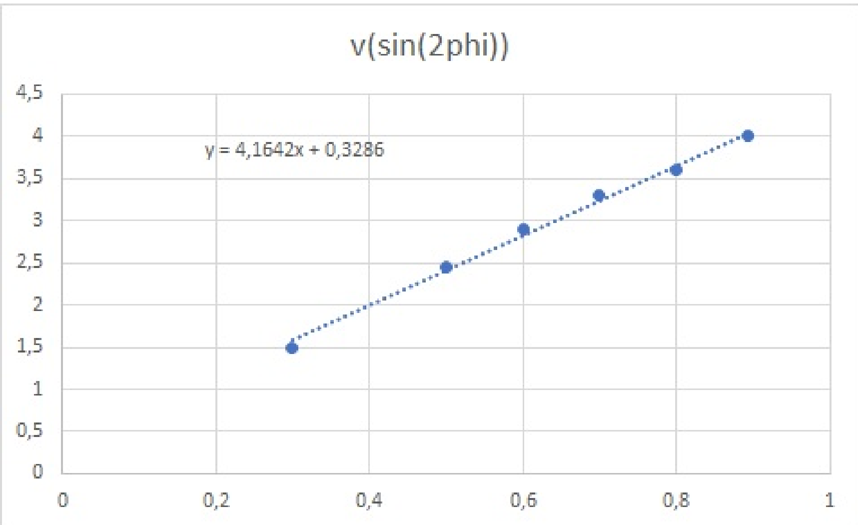



Its slope is

$$\boxed{\frac{\mathrm{d}v}{\mathrm{d}b} = 42}$$

**C4**. Plot the graph $v(\sin 2\varphi)$. Its slope is $v_c = 4.2$. At $b = b_{\text{cr}}$ the domain wall velocity becomes critical, so that

$$b_{\text{cr}} = \frac{v_c}{\mathrm{d}v/\mathrm{d}b} = 0.10.$$



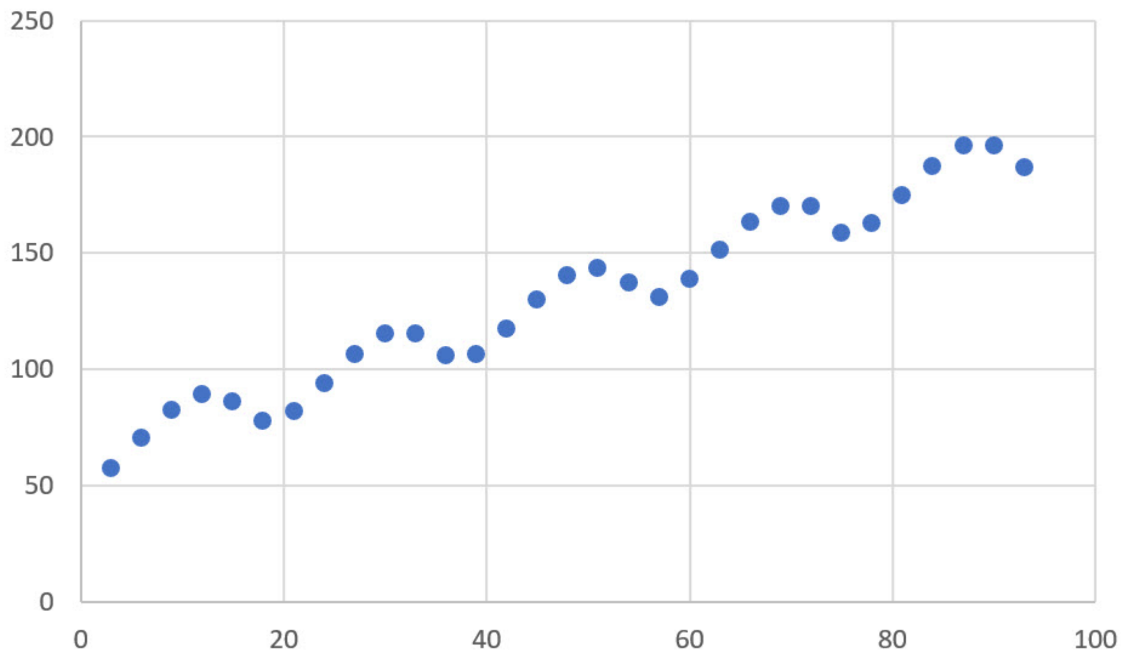

$$\boxed{v_c = 4.2, \qquad b_{\text{cr}} = 0.10}$$

**C5**. Answer

$$\boxed{v_c = 4.2\,a\omega = 1850\ \mathrm{m/s}; \qquad B_{\mathrm{cr}} = b_{\mathrm{cr}} B_{\mathrm{an}} = 0.083\ \mathrm{T}.}$$

**C6**. Run the program for $\alpha = 0.2$, $\beta = 100$, $b = 0.2$, $T = 100$, $L = 200$. Plot $m_z(x)$ for $t = 3, 6, 9, 12, \ldots, 90$ and determine $x_{\text{center}}$ as an $x$-coordinate of the intersection of the graph with the $x$-axis.

| $t$ | $x_{\text{center}}$ | $t$ | $x_{\text{center}}$ | $t$ | $x_{\text{center}}$ |
|---|---|---|---|---|---|
| 3 | 57.4 | 33 | 115.2 | 63 | 151.4 |
| 6 | 70.1 | 36 | 105.9 | 66 | 163.3 |
| 9 | 82.0 | 39 | 106.2 | 69 | 170.2 |
| 12 | 89.2 | 42 | 117.1 | 72 | 170.0 |
| 15 | 86.1 | 45 | 129.9 | 75 | 158.3 |
| 18 | 77.3 | 48 | 140.2 | 78 | 162.8 |
| 21 | 81.5 | 51 | 143.6 | 81 | 174.9 |
| 24 | 93.6 | 54 | 137.2 | 84 | 187.4 |
| 27 | 106.1 | 57 | 131.0 | 87 | 196.2 |
| 30 | 115.0 | 60 | 138.7 | 90 | 196.1 |




*The plot $x_{center}(t)$.*