6. Mechano-electrical oscillator

1) From the Newton's second law, $m\ddot{x} = -kx$, hence $\ddot{x} = -\frac{k}{m}x$, hence $\omega = \sqrt{k/m}$.

2) From the Gauss' law, the charge on the plate $Q = S\varepsilon_0 E = S\varepsilon_0 U/X_1$. The force acting on it $F_e = k(X_0 - X_1) = Q\langle E\rangle$, where $\langle E\rangle$ is the average electric field (averaged over the charges). Let us look at the charge layer (at the surface of the plate) with a high magnification: the electric field there depends linearly on the net charge inwards (in the plate) from the current point. Therefore, the average field is just the arithmetic average of the fields on both sides of the layer: $\langle E\rangle = E/2$. Finally, $F_e = k(X_0 - X_1) = QE/2$ (this result could have been obtained from energetic considerations, using infinitesimal virtual displacement of the plate and the energy conservation law). So,

$$F_e = \frac{S}{2}\varepsilon_0(U/X_1)^2, \quad \text{hence} \quad U = X_1\sqrt{2k(X_0 - X_1)/S\varepsilon_0}.$$

3) If the plates move by x, the change of the force due to electric field is

$$\delta F_e = x\left|\frac{d}{dX_1}\frac{S}{2}\varepsilon_0(U/X_1)^2\right| = \frac{x}{X_1}S\varepsilon_0(U/X_1)^2;$$

bearing in mind that $\frac{S}{2}\varepsilon_0(U/X_1)^2 = k(X_0 - X_1)$, we obtain $\delta F_e = 2\frac{x}{X_1}k(X_0 - X_1)$. There is also force cahnge due to elasticity: $\delta F_k = -kx$; the two forces have opposite sign (while approaching the discs, δF_k tries to push back, and δF_e tries to pull disks even closer). So,

$$\delta F = -kx\left[1 - 2\left(\frac{X_0}{X_1} - 1\right)\right] = -kx\left(3 - 2\frac{X_0}{X_1}\right).$$

Finally, $\ddot{x} = \delta F/m = -x\frac{k}{m}\left(3 - 2\frac{X_0}{X_1}\right)$, and

$$\omega = \sqrt{\frac{k}{m}\left(3 - 2\frac{X_0}{X_1}\right)}.$$

4) Now we have two oscillating variables, x and q. First, we write down the equation due to Kirchoff's laws:

$$L\ddot{q} = -\frac{q}{C} - xQ\frac{d}{dX_1}C^{-1}.$$

Here, the second term describes the voltage change on the capacitor due to the change of the capacitance (we approximate the real change by differential, valid for small shifts x). Note that $C^{-1} = X_1/S\varepsilon_0$ and $Q = S\varepsilon_0 U/X_1$; hence $\frac{d}{dX_1}C^{-1} = 1/S\varepsilon_0$, and

$$L\ddot{q} = -\frac{q}{C} - U\frac{x}{X_1}.$$

Here, the sign of the second term assumes that the x-axes is directed upwards (there is no current in the inductance and $L\ddot{q} = 0$, if the voltage on the capacitor keeps constant; for increasing charge $q > 0$, this assumes increasing capacitance, i.e. $x < 0$; in a full agreement with the signs of the above expression).

The second equation describes the Newton second law. First we note that the expression for F_e can be rewritten as $F_e = Q^2/2S\varepsilon_0$. So, if the charge on the plate does not change ($q = 0$), neither does change F_e. So, $\delta F_e = q\frac{d}{dQ}Q^2/2S\varepsilon_0 = qQ/S\varepsilon_0$. The infinitesimal force changes (δF_k and δF_e) can be simply added:

$$m\ddot{x} = -kx - qQ/S\varepsilon_0.$$

Now, let us look for a sinusoidal solution of circular frequency ω. Then, $\ddot{x} = -\omega^2 x$ and $\ddot{q} = -\omega^2 q$. Substituting this into the two above obtained equations, we find

$$\begin{cases}(L\omega^2 - C^{-1})q = xU/X_1\\(\omega^2 m - k)x = qQ/S\varepsilon_0\end{cases}.$$

This has a non-zero solution for x and q only if

$$(L\omega^2 - C^{-1})(\omega^2 m - k) = UQ/X_1 S\varepsilon_0.$$

Bearing in mind that $UQ/X_1 = 2k(X_0 - X_1)$ and $C = \varepsilon_0 S/X_1$, we can rewrite the equation as

$$(\varepsilon_0 SL\omega^2 - X_1)(\omega^2 m - k) = 2k(X_0 - X_1).$$

Introducing $\omega_0^2 = k/m$ and $\omega_1^2 = X_1/\varepsilon_0 SL$ we can further rewrite as

$$\omega^4 - \omega^2(\omega_1^2 + \omega_0^2) + \omega_0^2\omega_1^2\left(3 - 2\frac{X_0}{X_1}\right) = 0.$$

Therefore,

$$2\omega^2 = \omega_1^2 + \omega_0^2 \pm \sqrt{\omega_1^4 + \omega_0^4 + 2\omega_1^2\omega_0^2(X_0 X_1^{-1} - 5)},$$

i.e. this system has two eigenfrequencies, if $\frac{X_0}{X_1} < \frac{3}{2}$ (and becomes unstable, otherwise).