1. Spool ***(12 points)***

i) First solution. The momentary rotation centre of the spool is the contact point P with the floor (since this point is at rest). So, the velocity of the spool is $u' = R\omega$, where ω is the angular velocity. Consider triangle POA, where A is defined as the point where the loose end of the rope meets the inner part of the spool at the current moment of time, but which is actually a point of the spool, i.e. it rolls together with the spool); O is the centre of the spool. Let us denote $\angle PAO = \beta$; it is easy to see that $\angle AOP = \pi - \alpha$. The velocity $\vec{v}_A$ of the point A is perpendicular to PA and, hence, forms angle β with the loose end of the rope. Its projection to the rope equals to u, therefore $v_A = u/\cos\beta$. Further, $\omega = v_A/l$, where $l = |AP|$ can be found from the cosine theorem: $l = \sqrt{R^2 + r^2 + 2Rr\cos\alpha}$. The angle β can be found using the sine theorem for the triangle AOP: $\sin\beta = \frac{R\sin\alpha}{l}$. Combining everything together we end up with

$$u' = \frac{uR}{\sqrt{R^2\cos^2\alpha + r^2 + 2Rr\cos\alpha}} = \frac{uR}{|R\cos\alpha + r|}.$$

Second solution. Let us decompose the velocity $\vec{v}_A$ into two components: the tangential component (parallel to the rope) equals (by modulus) to u; let us denote the radial component as u_r. Since the distance between O and A is constant, the projection of the velocities of O and A to the line OA are equal:

$$u_r = v\sin\alpha \Rightarrow v = u_r/\sin\alpha \Rightarrow \omega = u_r/R\sin\alpha.$$

The vertical component of the velocity of the point A remains unchanged if we switch the laboratory system of reference with the system associated with point O; hence,

$$u\sin\alpha - u_r\cos\alpha = \omega r\sin\alpha = u_r r/R \Rightarrow$$

$$v = \frac{u_r}{\sin\alpha} = \frac{uR}{R\cos\alpha + r}.$$

ii) *(2 pts)* The easiest way to solve this part is to use the energy balance for infinitesimal displacement of the cylinder and apply the answer to the previous question:

$$Fu\cdot dt = d\left[\frac{M}{2}v^2\left(1 + \frac{J}{MR^2}\right)\right] = Mvdv\left(1 + \frac{J}{MR^2}\right) \Rightarrow$$

$$a = \frac{dv}{dt} = \frac{Fu}{Mv\left(1 + \frac{J}{MR^2}\right)} = \frac{F}{M}\cdot\frac{\cos\alpha + \frac{r}{R}}{1 + \frac{J}{MR^2}}.$$

iii) Let us write the force balance projection to the horizontal axis assuming that the spool is at the edge of slipping, i.e. the friction force $F_f = \mu_{\min}N$, where $N = mg - F\sin\alpha$ is the normal force:

$$Ma = F\cos\alpha + \mu_{\min}N = F\cos\alpha + \mu_{\min}(Mg - F\sin\alpha).$$

Using the result of the previous task, we can use this equation directly to obtain an expression for the minimal allowed value of the coefficient of friction:

$$\mu_{\min} = \frac{\left|\frac{r}{R} - \frac{J}{MR^2}\cos\alpha\right|}{\left(1 + \frac{J}{MR^2}\right)\left|\frac{Mg}{F} - \sin\alpha\right|}.$$

iv) The angular moment of the spool with respect to the edge of the threshold conserves during the impact (since the impact force has zero arm):

$$Mu(R - H) + J\frac{u}{R} = \left(J + MR^2\right)\frac{v}{R} \Rightarrow$$

$$v = u\left(1 - \frac{H/R}{1 + \frac{J}{MR^2}}\right).$$

v) From the energy conservation law we obtain immediately

$$\left(J + MR^2\right) \frac{v^2}{R^2} = \left(J + MR^2\right) \frac{w^2}{R^2} + 2MgH \Rightarrow$$

$$w = \sqrt{v^2 - \frac{2gH}{1 + \frac{J}{MR^2}}}.$$

vi) The spool is the most prone to jumping immediately after the impact; the gravity force needs to be large enough to bind the centre of mass to the rotational motion around the edge of the threshold:

$$\frac{Mv^2}{R} \leq g\frac{R-H}{R} \Rightarrow v^2 \leq \frac{g}{M}(R-H) \Rightarrow$$

$$u_0 = \sqrt{\frac{g}{M}(R-H)}\, \frac{1 + \frac{J}{MR^2}}{1 + \frac{J}{MR^2} - \frac{H}{R}}.$$