7. Vacuum bulb ***(8 points)***

i) Each pumping cycle reduces the number of molecules inside the bulb by a factor of $(1-\alpha)$; therefore, after N cycles, the number of molecules (and hence, the pressure) by a factor of $\beta = (1-\alpha)^N \approx e^{-N\alpha} \Rightarrow$

$$N = -\frac{\ln \beta}{\alpha}.$$

ii) Majority of the pumping cycles are done when the pressure inside the bulb is negligible as compared to the outside pressure. During such a cycle, a work equal to $p_0 V\alpha$ is done. Therefore,

$$A \approx N p_0 V \alpha = p_0 V |\ln \beta|.$$

iii) Due to adiabatic law, $pV^\gamma = \text{Const}$; when combined with the gas law $pV \propto T$ we obtain $p^{\gamma-1} \propto T^\gamma$. During the last downwards motion of the piston, the pressure inside the cylinder is increased by a factor of $1/\beta$; thus, $T = T_0 \beta^{\frac{1}{\gamma}-1}$.

iv) According to the modified pumping scheme, the work/energy loss is only due to the release of the hot air. Note that if we had a cylinder of volume V, we could be able to create vacuum inside there using only one pumping motion, i.e. by performing work $A = p_0 V$ and without any energy loss. Now, we perform an excess work, which is converted into internal energy of the released hot air, which needs to be calculated. Let $\xi = \frac{p}{p_0}$ be an intermediate rarefaction factor; then, we can apply the previous result to calculate the internal energy of released air, if its quantity is $d\nu$ moles: $dU = T_0\big(\xi^{\frac{1}{\gamma}-1} - 1\big) c_V \, d\nu$. Let us note that the number of moles inside the bulb is $\nu = \frac{p_0 \xi V}{R T_0} \Rightarrow d\nu = \frac{p_0 V}{R T_0} d\xi$. So,

$$U = p_0 V \frac{c_V}{R} \int_0^1 \big(\xi^{\frac{1}{\gamma}-1} - 1\big) d\xi = (\gamma - 1) p_0 V \frac{c_V}{R}.$$

Now, recall that $\gamma = c_p/c_V = 1 + \frac{R}{c_V}$, hence $\frac{c_V}{R} = \frac{1}{\gamma - 1}$ and $U = p_0 V$. This gives us the energy loss due to heating the released air; another $p_0 V$ is required for loss-free creation of the vacuum. Hence, the total required work is $A = 2 p_0 V$.