4. MAGNETIC DIPOLES (7 points)

i) *(3 points)* There is no torque on the square if $\theta = 0$ or $\theta = \pi$, so one of them is stable and the other unstable. If we start from $\theta = 0$ and turn the square to some θ, but keep two sides of the square perpendicular to $\vec{B}$, Lorentz forces on these two sides give a torque $\tau = -BIa \cdot a \sin\theta = -Bm \sin\theta$ towards decreasing θ. By symmetry, we get the same result if we keep the other keep two sides of the square perpendicular to $\vec{B}$. It is possible to conclude that the torque depends only on θ (at least near $\theta = 0$), not on the exact orientation of the square. Since torque acts to restore $\theta = 0$, we find that $\theta_s = 0$ and $\theta_u = \pi$. To find the work to get from θ_s to θ_u, we can again keep two sides perpendicular to $\vec{B}$ - the answer cannot depend on the path, so we choose the simplest one. Integrating $\tau = Bm \sin\theta$ from $\theta = 0$ to π gives us $w = 2Bm$.

ii) *(4 points)* Let us denote the number of electrons (per unit volume) with magnetic moment projection $+\mu_B$ as n_+ and the ones with $-\mu_B$ as n_-. Their sum is always the same, $n_+ + n_- = n$. Also, in thermal equilibrium, their ratio is given by $\frac{n_-}{n_+} = \exp\left(-\frac{2\mu_B B}{k_B T}\right)$, where k_B is Boltzmann's constant. Solving the equations, we can find n_+ and n_-. The total magnetic moment per unit volume (in the direction of $\vec{B}$) is given by $M = \mu_B(n_+ - n_-)$. After substituting,

$$M = \mu_B N \frac{1 - \exp\left(\frac{2\mu_B B}{k_B T}\right)}{1 + \exp\left(\frac{2\mu_B B}{k_B T}\right)} = \mu_B n \tanh\left(\frac{\mu_B B}{k_B T}\right).$$

Additional comments. We see B and M always have the same sign, therefore $\vec{M}$ is parallel with $\vec{B}$. This makes sense, as we saw that $\theta = 0$ ($\vec{m}$ parallel to $\vec{B}$) orientation had lowest energy. The graph of M vs B goes to $\mu_B n$ for very large B or to $-\mu_B n$ for very small B (all spins aligned with $\vec{B}$). At $B = 0$, $M = 0$ as well, since both spin orientations have the same energy. Around zero, the curve is linear, as $\tanh x \approx x$ for small x gives us

$$M \approx \frac{\mu_B^2 n B}{k_B T}.$$