8. ZENER DIODE (7 points)

i) *(1 point)* Kirchoff's 2nd law gives $L\dot{I} + \frac{q}{C} = 0$ or $\ddot{q} + \frac{1}{LC}q = 0$. This is the equation of a simple harmonic oscillator with frequency $\omega = \frac{1}{\sqrt{LC}}$ and we can immediately write $q(t) = q_0 \cos \omega t$, while $I(t) = \dot{q}(t) = -\omega q_0 \sin \omega t$.

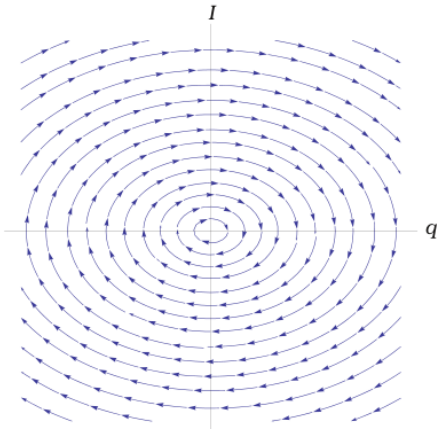
Note that

$$q^2 + \frac{1}{\omega^2} I^2 = q_0^2 (\sin^2 \omega t + \cos^2 \omega t) = q_0^2,$$

and therefore the phase diagram of the system is an ellipse centred at the origin, with semi-axes q_0 and ωq_0. Alternatively, this relation comes directly from the conservation of energy:

$$\frac{LI^2}{2} + \frac{q^2}{2C} = E_0 = \frac{q_0^2}{2C}.$$

By looking at q and I a quarter-period later from $t = 0$, say, it's not hard to see that the system must evolve in a clockwise sense on the phase diagram. Note that in this instance, only $q = 0$ is an equilibrium point: for all non-zero q there will be never-ending oscillations in the circuit.



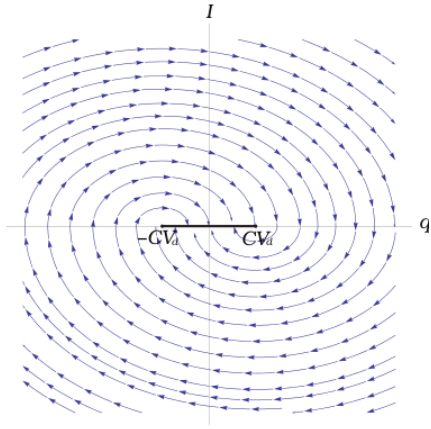
ii) *(2 points)* Now the sign of the voltage on the diode depends on the direction of the current, giving either of $L\ddot{q} + \frac{q}{C} \pm V_d = 0$. We can summarize the equations as follows:

$$L\ddot{q} + \frac{q}{C} = V_d \qquad \text{if } \dot{q} < 0$$
$$L\ddot{q} + \frac{q}{C} = -V_d \quad \text{if } \dot{q} > 0$$

Let us introduce the new variables $q_{1,2}$ such that $q_1 = q - CV_d$ and $q_2 = q + CV_d$. Then we can rewrite the two equations above in a more familiar form:

$$L\ddot{q}_1 + \frac{q_1}{C} = 0 \quad \text{if } \dot{q} < 0$$
$$L\ddot{q}_2 + \frac{q_2}{C} = 0 \quad \text{if } \dot{q} > 0$$

Thus the introduction of the diode only serves to shift the equilibrium points for the otherwise simple harmonic orbits. For $\dot{q} > 0$, the equilibrium point is $q_2 = 0$ or $q = -CV_d$, while for $\dot{q} < 0$ it is $q = CV_d$. So the orbit will consist of half-ellipses in the upper and the lower parts of the $I - q$ diagram, centred at $q = -CV_d$ for the upper half and at $q = CV_d$ for the lower half. As the evolution is continuous, these half-ellipses will join up at $I = 0$.



iii) *(2 points)* We can see on the diagram that there is a "dead zone" between $\pm CV_d$ (for $I = 0$). If a trajectory reaches any of the points in that segment, it will stay there forever. The extent of that region is $2CV_d$.

iv) *(2 points)* Let's use the phase diagram to figure this out. Suppose the capacitor initially has the charge $q_0 \gg CV_d$. Then the charge will first swing to the other way of CV_d and will become $q_{T/2} = CV_d - (q_0 - CV_d) = 2CV_d - q_0$. Then it will perform the other half-oscillation around $-CV_d$ and the charge at the end of that is $q_T = -CV_d + (-CV_d - (2CV_d - q_0)) = q_0 - 4CV_d$, and therefore $\Delta q = -4CV_d$.

Note that we have the right to talk about half- and full periods because the oscillations still happen at the immutable frequency $\omega = \frac{1}{\sqrt{LC}}$. Therefore the time between the two maxima is just a full period of oscillation, $T = \frac{2\pi}{\omega}$.

Once $q(t)$ has a zero derivative inside the region bounded by $\pm CV_d$, it will remain at that particular value forever. For a large initial q_0, we expect there to be approximately $\left|\frac{q_0}{\Delta q}\right| = \frac{|q_0|}{4CV_d}$ total oscillations.

More exactly, the distance from the "dead zone" is initially $|q_0| - CV_d$ and decreases during each half-oscillation by $2CV_d$. The total number of half-oscillations is $N = \left\lfloor \frac{|q_0| - CV_d}{2CV_d} \right\rfloor$ and the total time $t = N\frac{T}{2} = N\frac{\pi}{\omega} = N\pi\sqrt{LC}$.