7. Spin system

1) According to Boltzmann's distribution, $p_\uparrow = A \cdot e^{-\varepsilon m}$, where the constant A can be found from the condition that the probability of having either up or down orientation is one: $A \cdot e^{-\varepsilon/2} + A \cdot e^{\varepsilon/2} = 1$, hence

$$A = \frac{1}{e^{-\varepsilon/2kT} + e^{\varepsilon/2kT}} = \frac{1}{2\cosh(\varepsilon/2kT)}.$$

Thus,

$$p_\uparrow = \frac{e^{-\varepsilon/2kT}}{e^{-\varepsilon/2kT} + e^{\varepsilon/2kT}}.$$

2) The average energy is the weighted average of up- and down-state energies for a single spin, multiplied by the number of spins:

$$E = \frac{N\varepsilon}{2} \frac{e^{-\varepsilon/2kT} - e^{\varepsilon/2kT}}{e^{-\varepsilon/2kT} + e^{\varepsilon/2kT}} = -\frac{N\varepsilon}{2} \tanh(\varepsilon/2kT).$$

3) For small values of the argument of the hyperbolic tangent, the last expression can be approximated as $E \approx -N\varepsilon^2/4kT$.

4) According to the definition of the heat capacity, $C = \frac{dE}{dT} = N\varepsilon^2/4kT^2$.