2. HOLOGRAPHIC LENS

i) Let $N = 0, 1, \ldots$ number the zones (both opaque and transparent). The optical path difference between two neighbouring zones must be $\lambda/2$ (opposite phase is demanded). The path difference between the N^{th} zone and the 0^{th} zone, on the other hand, is $\Delta_N = \sqrt{r_N^2 + f^2} - f$. Therefore, $\frac{N\lambda}{2} = \sqrt{r_N^2 + f^2} - f$ and $r_N = \sqrt{\left(\frac{N\lambda}{2}\right)^2 + N\lambda f}$. Only odd-numbered zones are transparent, thus we need $r_{2m+1} = \sqrt{\left(m + \frac{1}{2}\right)^2 \lambda^2 + (2m+1)\lambda f}$.

ii) A perfectly focussing glass lens is such that all the possible light rays that go to the focus have an equal optical path length. The optical path length inside a refracting medium is n times longer than the corresponding geometric length (the phase velocity is slowed down by a factor of n). Denote the sought-after thickness by x. Equate the optical path lengths of a ray through the edge of the lens and of a ray through its centre: $\sqrt{\left(\frac{d}{2}\right)^2 + f^2} = f - x + nx \implies x = \frac{1}{n-1}\left[\sqrt{\left(\frac{d}{2}\right)^2 + f^2} - f\right] \approx 2.4\,\text{cm}$.

iii) Firstly, note that the given pulse is short enough that the whole lens never illuminates the focus — the pulse is only $\frac{\tau c}{\lambda} = 18$ periods long, but $r_{2\times 18} \approx 1.3\,\text{mm} \ll 5\,\text{cm}$. This implies that only a thin strip of the lens is illuminating the focus at a time. The intensity, when the N^{th} period is being observed, is proportional to the area of the N^{th} zone. This is $A_N = \pi\left(r_{N+1}^2 - r_N^2\right) = \pi\left(\frac{N\lambda^2}{2} + \frac{\lambda^2}{4} + \lambda f\right)$.

As N is proportional to time (the period of the wave is constant), the intensity will also grow linearly in time. The linear part starts at $N = 1$ with a jump and ends at $N_{\max}\frac{\lambda}{2} = \sqrt{\left(\frac{d}{2}\right)^2 + f^2} - f$ with a jump back into darkness, when the light from the edge of the lens arrives. The total duration of illumination is (approximately) $\tau_{\text{hol}} = N_{\max}\frac{\lambda}{c} = \frac{2}{c}\left[\sqrt{\left(\frac{d}{2}\right)^2 + f^2} - f\right] \approx 7.9 \times 10^{-9}\,\text{s}$.

iv) The pulse is localized into a region of space with a width $\Delta x = c\tau$. Because of the Heisenberg's uncertainty principle, the pulse is composed of photons with a range of momenta (if we take the picture that the properties of the individual photons are classical) or, from a different viewpoint, is a single photon with a somewhat uncertain momentum; either way, the characteristic width in the momentum space is $\Delta p = \frac{\hbar}{\Delta x} = \frac{\hbar}{c\tau}$. The wavelength of a photon, whose momentum p is known, is $\lambda = \frac{h}{p}$ (this is the de Broglie relation: the photon's energy is pc and also $h\nu = \frac{hc}{\lambda}$). Thus, $\Delta\lambda \approx \left|\frac{\mathrm{d}}{\mathrm{d}p}\frac{h}{p}\right|\Delta p = \frac{h\Delta p}{p^2} = \frac{h\frac{\hbar}{c\tau}}{\left(\frac{h}{\lambda}\right)^2} = \frac{\lambda^2}{2\pi c\tau} \approx 4.4 \times 10^{-9}\,\text{m}$.

v) The spread of the arrival times of waves with different wavelengths is the largest for the waves that spend the longest time inside the lens. Therefore it is enough to consider only the waves that go through the thickest part of the lens — its centre. The spread in the arrival times is $\Delta t = \Delta\frac{x}{v_g} = \frac{x\Delta v_g}{v_g^2} = \frac{x\Delta\lambda}{v_g^2}\frac{\mathrm{d}v_g}{\mathrm{d}\lambda}$. To find the group velocity v_g itself, we can use the hint (given during the examination) that in this question we may assume the group velocity and the phase velocity to be equal (in reality it would be an unusual coincidence): $v_g = v_p$ and the phase velocity $v_p = c/n$. Therefore, $\Delta t = \frac{x\Delta\lambda}{v_g^2} \times 0.02\frac{v_g}{\lambda} = 0.02\frac{xn\Delta\lambda}{c\lambda} \approx 2.1 \times 10^{-14}\,\text{s}$. The total observed pulse length behind the glass lens is $\tau_{\text{gl}} = \tau + \Delta t = 5.1 \times 10^{-14}\,\text{s}$.